

Z-transforms : Part II

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1 Key Properties

From the relationship between the DTFT and the Z-transform we might expect that many of the DTFT properties carry over. We're right! Most of the properties can be derived with a line or two of algebra. Here are a few useful ones:

Linearity:

$$\begin{aligned} \text{Z-transform } \{ax[n] + by[n]\} &= \sum_{n=-\infty}^{\infty} (ax[n] + by[n])z^{-n} \\ &= a \sum_{n=-\infty}^{\infty} x[n]z^{-n} + b \sum_{n=-\infty}^{\infty} y[n]z^{-n} \end{aligned}$$

$$\boxed{ax[n] + by[n] \xleftrightarrow{Z} aX(z) + bY(z)}$$

Differentiation:

$$\begin{aligned} \frac{dX(z)}{dz} &= - \sum_{n=-\infty}^{\infty} x[n]nz^{-n-1} \\ &= -z^{-1} \sum_{n=-\infty}^{\infty} (nx[n])z^{-n} \end{aligned}$$

$$\boxed{nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}}$$

Shifting:

$$\begin{aligned} \text{Z-transform } \{x[n + n_0]\} &= \sum_{n=-\infty}^{\infty} x[n + n_0]z^{-n} \\ &= \sum_{p=-\infty}^{\infty} x[p]z^{-p+n_0} \\ &= z^{n_0} \sum_{p=-\infty}^{\infty} x[p]z^{-p} \end{aligned}$$

$$\boxed{x[n + n_0] \xleftrightarrow{Z} z^{n_0} X(z)}$$

When you are in doubt about how to apply properties from a table, then try deriving the Z -transform directly from your sequence. The formula is usually easy to simplify given $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, $|\alpha| < 1$.

2 Inverse Z -transform

The goal of an inverse Z -transform is to get $x[n]$ given $X(z)$. Here are four ways to find an inverse Z -transform, ordered by typical use:

1. Use a Z -transform table. (See additional handouts.)
2. Partial fraction expansion.
3. Long division.
4. Contour integration.

Using the tables can be easiest, but they are not always with you when you need them, and sometimes the case you want isn't in a table.

Partial fraction expansion and long division are the two simplest methods. Contour integration can almost always be avoided. You are not responsible for knowing how to do contour integration, but we can show you how it works outside of class if you're interested. However, it is good to have at least seen a contour integral so that you can recognize it if you see it again. Briefly, imagine C to be a closed contour that surrounds the origin of the z -plane and lies in the region of convergence for some $X(z)$. Then:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz,$$

where the integration is counterclockwise over C . There are simple ways to evaluate this integral for most of the functions we'll deal with in this class, but we'll leave this for outside discussions, as you will not be responsible for knowing this method.

In the rest of these notes we will work examples using partial fraction expansions and long division. For both cases, it does not matter whether you prefer z^{-1} 's or z 's, but always do the following first:

Make sure the fraction contains only nonpositive or only nonnegative powers of z .

Here's a quick example:

$$\frac{1 + az^{-1} + bz^{-2}}{cz + d + ez^{-1}} = \frac{z^{-1} + az^{-2} + bz^{-3}}{c + dz^{-1} + ez^{-2}} \quad \text{or} \quad \frac{z^2 + az + b}{cz^3 + dz^2 + ez}$$

If you try to work these methods without doing this, you are exceedingly likely to make mistakes. Many a quiz point has been lost by a student who forgets to do this step first. You have been warned.

3 Partial Fraction Expansion

For this method, we assume you know that both

$$\frac{1}{1 - az^{-1}} \quad \text{and} \quad \frac{z}{z - a} \tag{1}$$

transform either to $a^n u[n]$ or to $-a^n u[-n - 1]$ depending on whether $|z| > |a|$ or $|z| < |a|$ respectively.

To find the inverse Z -transform using partial fraction expansion there are two steps:

1. Convert the Z -transform to the form of a proper fraction.
2. Convert the proper fraction to a linear combination of expressions like (1).

Example: Find $h[n]$ by partial fraction expansion, given

$$H(z) = \frac{z^{-1} + z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$

First, let's make it proper. We can do this by dividing until it is a proper fraction, or by factoring out some z 's or z^{-1} 's and later applying the shifting property. The latter is usually easier:

$$\begin{aligned} zH(z) = H_1(z) &= \frac{1 + z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \\ &= \frac{1 + z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} \\ &= \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{2}z^{-1}}, \end{aligned}$$

where we need to find the constants A and B . Cross multiplying and dividing gives

$$A[1 - \frac{1}{2}z^{-1}] + B[1 - \frac{1}{4}z^{-1}] = 1 + z^{-1}.$$

Equating powers of z gives the two equations:

$$\begin{aligned} A + B &= 1 \\ -\frac{A}{2} - \frac{B}{4} &= 1. \end{aligned}$$

The solutions are $A = -5$, $B = 6$, which we plug back in. Then we can recognize the inverse transform:

$$\begin{aligned} H_1(z) &= \frac{-5}{1 - \frac{1}{4}z^{-1}} + \frac{6}{1 - \frac{1}{2}z^{-1}} \\ \xleftarrow{z} & -5\left(\frac{1}{4}\right)^n u[n] + 6\left(\frac{1}{2}\right)^n u[n]. \end{aligned}$$

Applying shifting yields:

$$\begin{aligned} H(z) &= z^{-1}H_1(z) \\ H(z) &\xleftarrow{z} -5\left(\frac{1}{4}\right)^{n-1} u[n - 1] + 6\left(\frac{1}{2}\right)^{n-1} u[n - 1]. \end{aligned}$$

How did we know to pick the right-sided inverse transforms as opposed to the left-sided ones or some mixture of both? We didn't; in fact they may not be the correct choice. We can't know which one is correct yet – we need more information first.

Let's back up and consider all the combinations for inverting $H_1(z)$ (before shifting):

$$\begin{array}{ll}
 -5\left(\frac{1}{4}\right)^n u[n] + 6\left(\frac{1}{2}\right)^n u[n] & \text{right-sided} \\
 5\left(\frac{1}{4}\right)^n u[-n-1] + -6\left(\frac{1}{2}\right)^n u[-n-1] & \text{left-sided} \\
 -5\left(\frac{1}{4}\right)^n u[n] + -6\left(\frac{1}{2}\right)^n u[-n-1] & \text{two-sided} \\
 5\left(\frac{1}{4}\right)^n u[-n-1] + 6\left(\frac{1}{2}\right)^n u[n] & \text{impossible}
 \end{array}$$

Since there are two poles in this system ($z = \frac{1}{4}, \frac{1}{2}$), there are three ways the ROC's can have a non-empty intersection (and a fourth case where there is no intersection). These four cases correspond to the ROC's:

$$\begin{array}{ll}
 \frac{1}{2} < |z| & \text{right-sided} \\
 |z| < \frac{1}{4} & \text{left-sided} \\
 \frac{1}{4} < |z| < \frac{1}{2} & \text{two-sided} \\
 |z| < \frac{1}{4} \quad \text{and} \quad \frac{1}{2} < |z| & \text{(no intersection!)}
 \end{array}$$

To know which of three possible answers it is, you have to know if the signal is left-, right-, or two-sided. Knowing it is absolutely summable would also give the answer uniquely. In general for $p \geq 2$ poles, the ROC is one of $p + 1$ cases: the region inside the innermost pole, the region outside the outermost pole, or one of the regions between two poles. Don't forget: the ROC can never contain a pole.

4 Long Division

Consider finding $h[n]$ by long division, given

$$H(z) = \frac{1}{1 - az^{-1}}.$$

$$\begin{array}{r}
 1 + az^{-1} + a^2z^{-2} + \dots \\
 \hline
 1 - az^{-1} \bigg) 1 \\
 \underline{1 - az^{-1}} \\
 az^{-1} \\
 \underline{az^{-1} - a^2z^{-2}} \\
 a^2z^{-2} \\
 \dots
 \end{array}$$

Division gives the result, $h[n] = a^n u[n]$ which converges if $|a| < 1$. But what if $|a| > 1$? Consider the same division, “flipped around”:

$$\begin{array}{r}
 -a^{-1}z \quad -a^{-2}z^2 - \dots \\
 \hline
 -az^{-1} + 1 \bigg) \quad 1 \\
 \phantom{-az^{-1} + 1 \bigg) \quad} 1 \quad -a^{-1}z \\
 \phantom{-az^{-1} + 1 \bigg) \quad} \underline{ \phantom{-a^{-1}z}} \\
 \phantom{-az^{-1} + 1 \bigg) \quad} \phantom{-a^{-1}z} a^{-1}z \\
 \phantom{-az^{-1} + 1 \bigg) \quad} \phantom{-a^{-1}z} \underline{a^{-1}z - a^{-2}z^2} \\
 \phantom{-az^{-1} + 1 \bigg) \quad} \phantom{-a^{-1}z} \phantom{a^{-1}z - a^{-2}z^2} a^{-2}z^2 \\
 \phantom{-az^{-1} + 1 \bigg) \quad} \phantom{-a^{-1}z} \phantom{a^{-1}z - a^{-2}z^2} \phantom{a^{-2}z^2} \dots
 \end{array}$$

Flipping the divisor gives a different result, $h[n] = -a^n u[-n - 1]$ which converges if $|a| > 1$.

Beware how you execute long division! For starters, make sure you have converted the numerator and denominator to be either all in powers of z^{-1} or all in powers of z , but not a mixture of both. Then it may be easier to split the fraction into a sum of fractions, each having a single term in the numerator as in the example above. Then be careful how you divide – try both of the above ways if you are unsure, and look at whether the coefficients are growing or shrinking. In general the method of long division is simple for one-sided sequences; however, it can lead to confusion if you are not careful at each step.

When in doubt, try working the same problem using two or more methods – it is a good way to catch those inevitable algebra mistakes, since the different methods should all lead to the same result.