Solving impulse response

$$y[n] = x[n] - y[n-2]$$

$$x[n] = \delta[n]$$

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$
$$y[1] = x[1] - y[-1] = 0 - 0 = 0$$
$$y[2] = x[2] - y[0] = 0 - 1 = -1$$
$$y[3] = x[3] - y[1] = 0 - 0 = 0$$
$$y[4] = x[4] - y[2] = 0 - (-1) = 1$$
$$\vdots$$

z-transform

$$Y(z) = X(z) - z^{-2}Y(z)$$

$$\left(1 + z^{-2}\right)Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 + z^{-2}\right)}$$

system function

$$Y(z) = H(z)X(z) = \frac{1}{\left(1 + z^{-2}\right)}X(z)$$

Remember:

$$X(z) = \frac{z^2 - \cos(\hat{\omega})z}{z^2 - 2\cos(\hat{\omega})z + 1}$$

$$\Updownarrow$$

$$x[n] = \cos(\hat{\omega}n)u[n]$$

$$\hat{\omega} = \frac{\pi}{2} = \frac{2\pi}{4}$$

$$Y(z) = \frac{1}{\left(1 + z^{-2}\right)} = \frac{z^2}{z^2 + 1}$$

$\Downarrow$ Inverse z-transform (lookup)

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n]$$

$$y[n] = \{1, 0, -1, 0, 1, \ldots\}$$

$$y[n] = x[n] - y[n-2]$$

The signal has same z-transform as system. The signal is the impulse response of the system

$\Downarrow$ z-transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1+z^{-2}\right)} = \frac{z^2}{z^2+1}$$

system function

$$zeros = roots(z^2) = 0{,}0$$
$$poles = roots(z^2+1) = \pm j$$

Poles: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \to \infty$

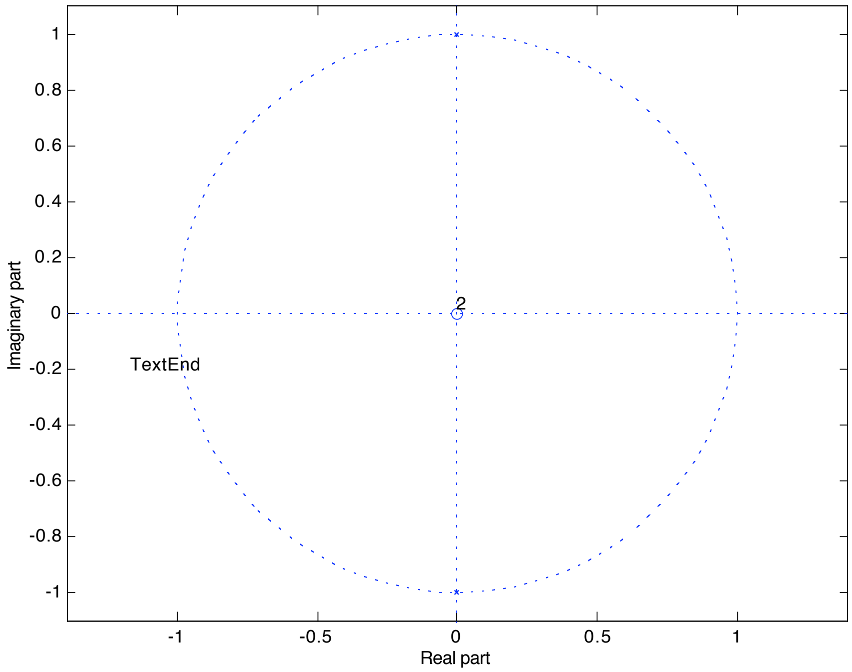
Zeros: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \to 0$

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n]$$

signal

$\Downarrow$ z-transform

$$Y(z) = \frac{1}{\left(1+z^{-2}\right)} = \frac{z^2}{z^2+1}$$



Poles: z locations of $y[n] = z^n$

Zeros: related to the magnitude and phase of $y[n] = z^n$

The closer a zero is to a pole, the smaller the effect the pole.

Solve difference equation

$$y[n] = x[n] - y[n-2] \qquad h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n]$$ impulse response

$$x[n] = u[n]$$ step input

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$
$$y[1] = x[1] - y[-1] = 1 - 0 = 1$$
$$y[2] = x[2] - y[0] = 1 - 1 = 0$$
$$y[3] = x[3] - y[1] = 1 - 1 = 0$$
$$y[4] = x[4] - y[2] = 1 - 0 = 1$$
$$\vdots$$

convolution

$$\begin{array}{lcccccccc}
\text{x[n]} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\text{h[n]} & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \ldots \\
\hline\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & & & -1 & -1 & -1 & -1 & -1 & \ldots \\
 & & & & 0 & 0 & 0 & 0 & \\
 & & & & & 1 & 1 & 1 & \\
 & & & & & & \vdots & & \\
\hline\hline
 & 1 & 1 & 0 & 0 & 1 & \ldots & &
\end{array}$$

z-transforms

$$y[n]=x[n]-y[n-2]$$

$$\Downarrow z$$

$$Y(z)=X(z)-z^{-2}Y(z)$$

$$Y(z)=\frac{1}{\left(1+z^{-2}\right)}X(z)=H(z)X(z)$$

$$x[n]=u[n]$$

$$\Downarrow z$$

$$X(z)=\frac{1}{1-z^{-1}}$$

multiplication

$$Y(z)=\frac{1}{\left(1+z^{-2}\right)}\frac{1}{\left(1-z^{-1}\right)}$$

$$\Downarrow$$ partial fraction expansion (distinct complex roots)

$$Y(z)=\frac{1/2}{\left(1-z^{-1}\right)}+\frac{1/2}{\left(1+z^{-2}\right)}+\frac{1}{2}\frac{z^{-1}}{\left(1+z^{-2}\right)}$$

$$\Downarrow$$ Inverse z-transform (lookup)

$$y[n]=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi}{4}n\right)+\frac{1}{2}\cos\left(\frac{2\pi}{4}\left(n-1\right)\right)\quad\Rightarrow$$

$y[n]$		$Y(z)$
$\cos\left(\frac{2\pi}{4}n\right)u[n]$	$\Leftrightarrow$	$\frac{1}{\left(1+z^{-2}\right)}$
$a^n u[n]$	$\Leftrightarrow$	$\frac{1}{\left(1-z^{-1}\right)}$

n	$y[n]$
0	1/2+1/2+0=1
1	1/2+0+1/2=1
2	1/2-1/2+0=0
3	1/2+0-1/2=0
4	1/2+1/2+0=1
5	1/2+0+1/2=1

Partial fraction expansion

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)} = \frac{A}{\left(1+2z^{-1}\right)} + \frac{B}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$\Downarrow$ cross multiply

$$Y(z) = \frac{A\left(1-\tfrac{3}{4}z^{-1}\right)+B\left(1+2z^{-1}\right)}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$\Downarrow$ collect terms

$$Y(z) = \frac{(-\tfrac{3}{4}A+2B)z^{-1}+\left(A+B\right)}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{A}{\left(1+2z^{-1}\right)} + \frac{B}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{(-\tfrac{3}{4}A+2B)z^{-1} + \left(A+B\right)}{\left(1+2z^{-1}\right)\left(1-z^{-1}\right)}$$

$-\tfrac{3}{4}A + 2B = 0$ $\quad A + B = 1$ match coefficients

$$A = \frac{8}{11} \qquad B = \frac{3}{11}$$

$$Y(z) = \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)} = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Inverse z-transform

$$Y(z)=\frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$=\frac{8/11}{\left(1+2z^{-1}\right)}+\frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

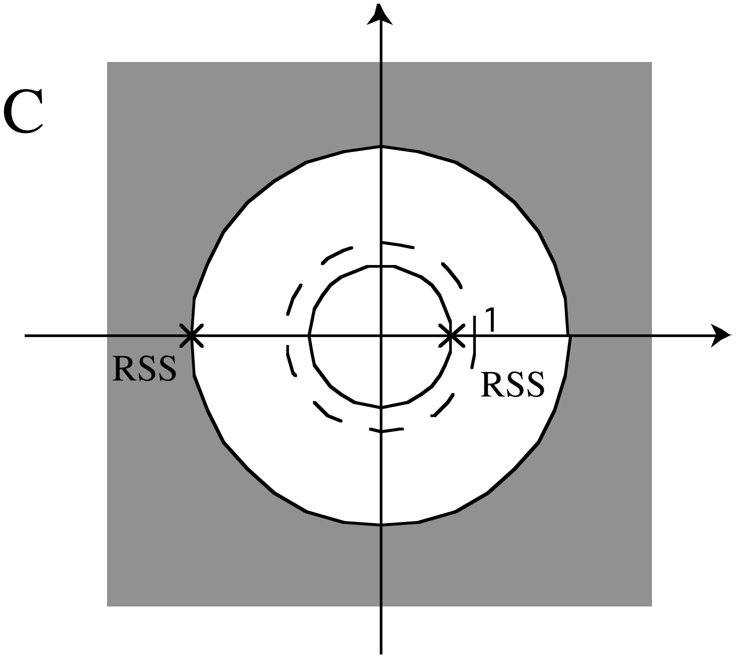
poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n]=\frac{8}{11}(-2)^{n}u[n]+\frac{3}{11}\left(\frac{3}{4}\right)^{n}u[n] \qquad |z|>|2|$$

causal, unstable

ROC



We know:

right sided sequence

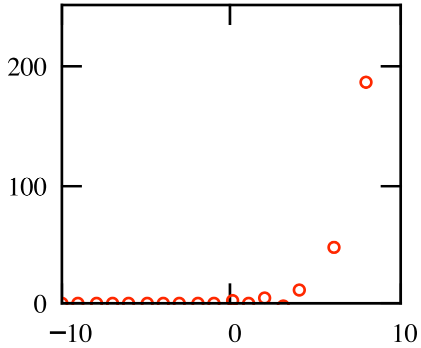
$$\frac{1}{1-az^{-1}}\Leftrightarrow a^{n}u[n]$$

$$|z|>|a|$$

left sided sequence

$$\frac{1}{1-az^{-1}}\Leftrightarrow -a^{n}u[-n-1]$$

$$|z|<|a|$$



Inverse z-transform

$$Y(z)=\frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

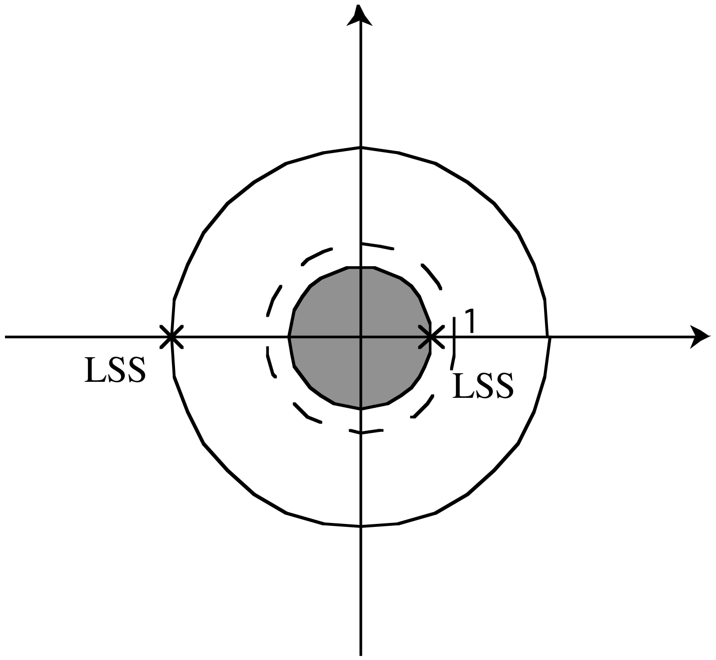
$$=\frac{8/11}{\left(1+2z^{-1}\right)}+\frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n]=-\frac{8}{11}(-2)^{n}u[-n-1]-\frac{3}{11}\left(\frac{3}{4}\right)^{n}u[-n-1] \qquad |z|<\left|\tfrac{3}{4}\right|$$

anticausal, unstable



We know:

right sided sequence

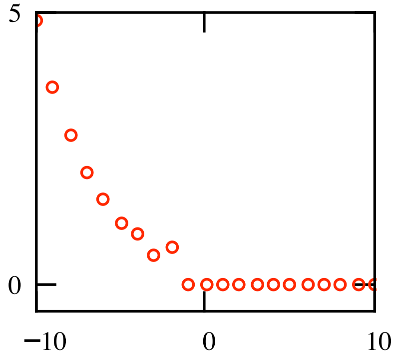
$$\frac{1}{1-az^{-1}}\Leftrightarrow a^{n}u[n]$$

$$|z|>|a|$$

left sided sequence

$$\frac{1}{1-az^{-1}}\Leftrightarrow -a^{n}u[-n-1]$$

$$|z|<|a|$$



Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

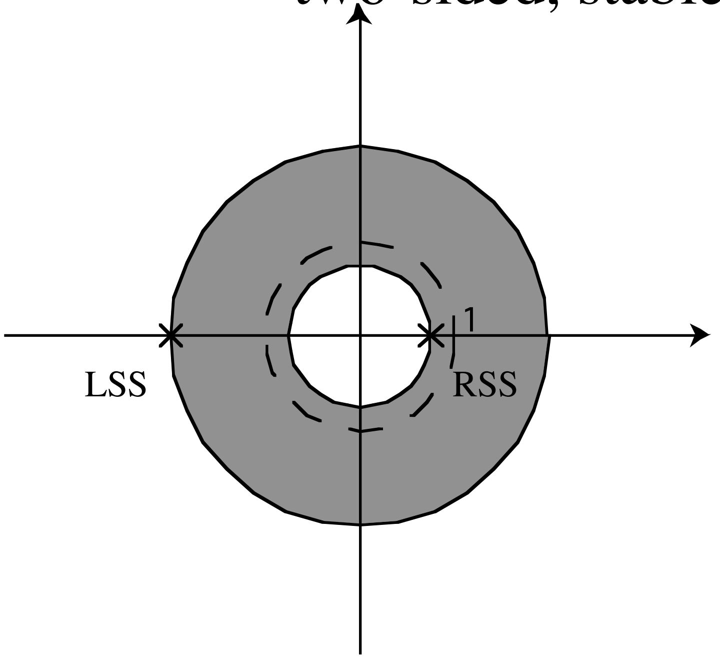
poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$

two-sided, stable



We know:

right sided sequence

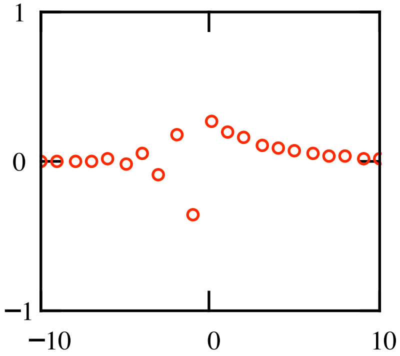
$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$



Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\frac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$\Downarrow$ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1]$$

$$\left|\tfrac{3}{4}\right| < |z| \cap |z| > |2|$$

not possible

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Inverse z-transform

$$Y(z) = \frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1+2z^{-1}\right)} + \frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$\Downarrow$ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \qquad |z| > |2|$$

causal, unstable

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \qquad |z| < \left|\tfrac{3}{4}\right|$$

anticausal, unstable

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \qquad \left|\tfrac{3}{4}\right| < |z| < |2|$$

two-sided, stable

$$y[n] = \frac{8}{11}(-2)^n u[n] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \qquad \left|\tfrac{3}{4}\right| < |z| \cap |z| > |2|$$

not possible

Partial fraction expansion II

$$Y(z) = \frac{4+7.6z^{-1}}{-6z^{-2}+5z^{-1}+4} = \frac{4+7.6z^{-1}}{\left(1+2z^{-1}\right)\left(4-3z^{-1}\right)}$$

$$= \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

factor into form

$$\left(1+p_1z^{-1}\right)\left(1-p_2z^{-1}\right)$$

$$= \frac{A}{\left(1+2z^{-1}\right)} + \frac{B}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$A = Y(z)\left(1+2z^{-1}\right)\Big|_{z=-2} = \left.\frac{1+1.9z^{-1}}{\left(1-\tfrac{3}{4}z^{-1}\right)}\right|_{z=-2} = 0.036$$

$$B = Y(z)\left(1-\tfrac{3}{4}z^{-1}\right)\Big|_{z=\frac{3}{4}} = \left.\frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)}\right|_{z=\frac{3}{4}} = 0.964$$

$$Y(z) = \frac{0.036}{\left(1+2z^{-1}\right)} + \frac{0.964}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1-az^{-1}} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

Effects of a zero

zeros: 0,-1.9

$$Y(z)=\frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

poles:3/4,-2

$$Y(z)=\frac{0.036}{\left(1+2z^{-1}\right)}+\frac{0.964}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$y[n]=-0.036(-2)^{n}u[-n-1]+0.964\left(\tfrac{3}{4}\right)^{n}u[n]$$

$$\left|\tfrac{3}{4}\right|<|z|<|2|$$

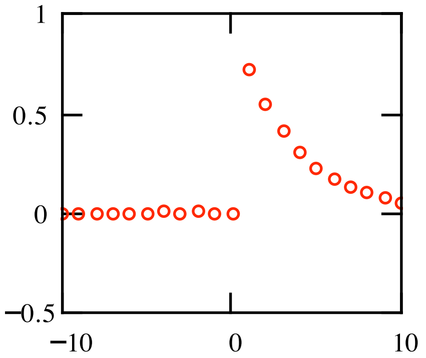
zeros: 0,0

$$Y(z)=\frac{1}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

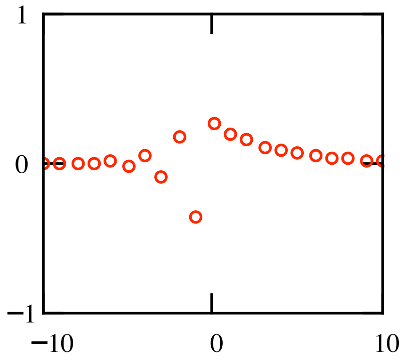
$$=\frac{8/11}{\left(1+2z^{-1}\right)}+\frac{3/11}{\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$y[n]=-\frac{8}{11}(-2)^{n}u[-n-1]+\frac{3}{11}\left(\frac{3}{4}\right)^{n}u[n]$$

$$\left|\tfrac{3}{4}\right|<|z|<|2|$$



zero at z=-1.9
close to pole at z=-2,
pole's effect reduced
(0.036 vs. 0.727)



two sided sequences

Partial fraction expansion III

$$Y(z)=\frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$=\frac{4+7.6z^{-1}}{-6z^{-2}+5z^{-1}+4}$$

$$=\frac{4z^2+7.6z}{4z^2+5z-6}$$

$$=\frac{z^2+1.9z}{(z+2)\left(z-\tfrac{3}{4}\right)}$$

factor into form (z-p1)(z-p2)
Hint: use matlab’s **root** command

$$=\frac{A}{(z+2)}+\frac{B}{\left(z-\tfrac{3}{4}\right)}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}}=\frac{z}{z-a}\Leftrightarrow a^n u[n]$$

$$|z|>|a|$$

left sided sequence

$$\frac{1}{1-az^{-1}}=\frac{z}{z-a}\Leftrightarrow -a^n u[-n-1]$$

$$|z|<|a|$$

Partial fraction expansion III

$$Y(z)=\frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\tfrac{3}{4}z^{-1}\right)}$$

$$=\frac{4z^2+7.6z}{4z^2+5z-6} \quad =\frac{z^2+1.9z}{(z+2)\left(z-\tfrac{3}{4}\right)}$$

$$=\frac{Az}{(z+2)}+\frac{Bz}{\left(z-\tfrac{3}{4}\right)}+C$$

$$C=\left.\frac{4z^2+7.6z}{4z^2+5z-6}\right|_{z=0}=0$$

$$B=\left.\frac{Y(z)(z-\tfrac{3}{4})}{z}\right|_{z=\frac{3}{4}}=\left.\frac{z^2+1.9z}{z(z+2)}\right|_{z=\frac{3}{4}}=0.964$$

$$A=\left.\frac{Y(z)(z+2)}{z}\right|_{z=-2}=\left.\frac{z^2+1.9z}{z(z-\tfrac{3}{4})}\right|_{z=-2}=0.036$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}}=\frac{z}{z-a}\Leftrightarrow a^n u[n]$$

$$|z|>|a|$$

left sided sequence

$$\frac{1}{1-az^{-1}}=\frac{z}{z-a}\Leftrightarrow -a^n u[-n-1]$$

$$|z|<|a|$$

Partial fraction expansion III

$$Y(z) = \frac{1 + 1.9z^{-1}}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

$$= \frac{4z^2 + 7.6z}{4z^2 + 5z - 6}$$

$$= \frac{0.036z}{(z+2)} + \frac{0.964z}{\left(z - \frac{3}{4}\right)}$$

$$y[n] = -0.036(-2)^n u[-n-1] + 0.964\left(\tfrac{3}{4}\right)^n u[n]$$

$$\left|\tfrac{3}{4}\right| < |z| < |2|$$

We know:

right sided sequence

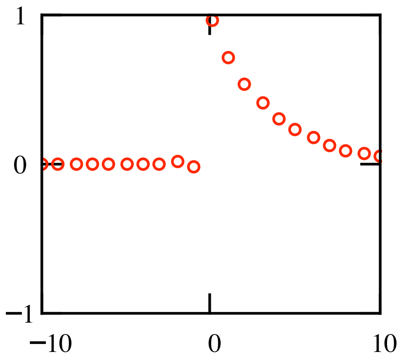
$$\frac{1}{1 - az^{-1}} = \frac{z}{z-a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z-a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$



two sided sequence

Long Division

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} = \frac{1+1.9z^{-1}}{-\frac{3}{2}z^{-2}+\frac{5}{4}z^{-1}+1}$$

$$1+0.65z^{-1}+0.687z^{-2}+0.116z^{-3}$$

right sided sequence

$$= 1+\frac{5}{4}z^{-1}-\frac{3}{2}z^{-2}\,\overline{\big)\,1+1.9z^{-1}}$$

$$\underline{1+1.25z^{-1}-1.5z^{-2}}$$

$$0.65z^{-1}+1.5z^{-2}$$

$$\underline{0.65z^{-1}+0.813z^{-2}-0.975z^{-3}}$$

$$0.687z^{-2}+0.975z^{-3}$$

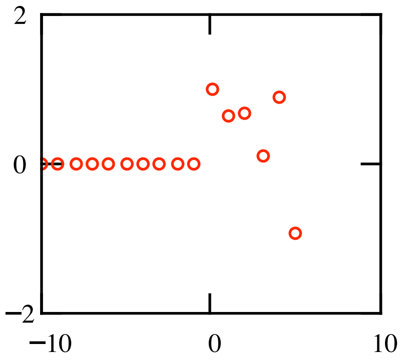
$$\underline{0.687z^{-2}+0.859z^{-3}-1.031z^{-4}}$$

$$0.116z^{-3}+1.031z^{-4}$$

$$y[n] = \delta[n]+0.65\delta[n-1]+0.69\delta[n-2]+0.12\delta[n-3]+\cdots$$

compare

$y[n] = 0.036\left(-2\right)^{n}u[n]+\left(\frac{3}{4}\right)^{n}0.964u[n]$ ={1, 0.65, 0.69, 0.12} n=0,1,2,3



Long Division

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)} = \frac{1+1.9z^{-1}}{-\frac{3}{2}z^{-2}+\frac{5}{4}z^{-1}+1}$$

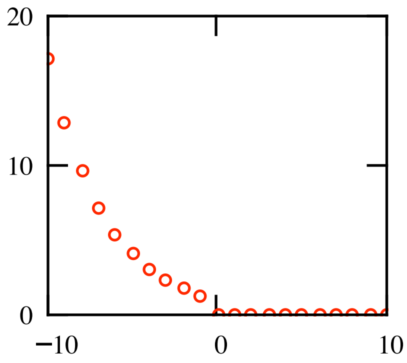
left sided sequence

$$\begin{array}{r l}
 & -1.267z-1.723z^{2}-2.281z^{3}+\ldots \\
= -1.5z^{-2}+1.25z^{-1}+1 \big) & 1.9z^{-1}+1 \\
 & \underline{1.9z^{-1}-1.584-1.267z} \\
 & 2.584+1.267z \\
 & \underline{2.584-2.154z-1.723z^{-3}} \\
 & 3.421z^{-2}+0.975z^{-3} \\
 & \underline{3.421z^{-2}-2.851z^{-3}-2.281z^{-4}}
\end{array}$$

$$y[n] = -1.27\delta[n+1]-1.72\delta[n+2]-2.28\delta[n+3]+\cdots$$

compare

$y[n] = -0.036\left(-2\right)^{n}u[-n-1]-\left(\frac{3}{4}\right)^{n}0.964u[-n-1]$={-1.27, -1.72, -2.28}
n={-1, -2, -3}



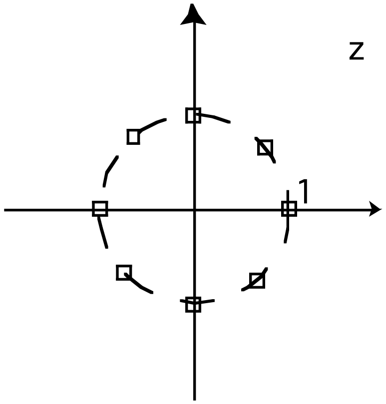
Discrete Fourier Transform (DFT)

Compute spectrum of discrete-time periodic signals

N samples in time domain $\underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}}$ N complex numbers in frequency domain

DFT $$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi k/N)n}$$ analysis

IDFT $$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j(2\pi k/N)n}$$ synthesis



DFT: sample continuous H(ω) at N evenly spaced frequencies

Discrete Fourier Transform (DFT)

Compute spectrum of discrete-time periodic signals

N samples in time domain $\underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}}$ N complex numbers in frequency domain

Sampling $x[n] = x(nT_s)$

Discrete time
Periodic freq

Fourier Series $$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt$$

Periodic in time
Discrete freq

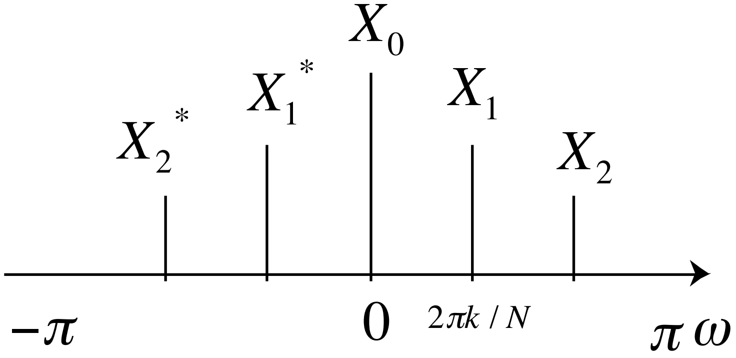
DFT $$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi k/N)n}$$

Discrete & periodic time
Discrete & periodic freq

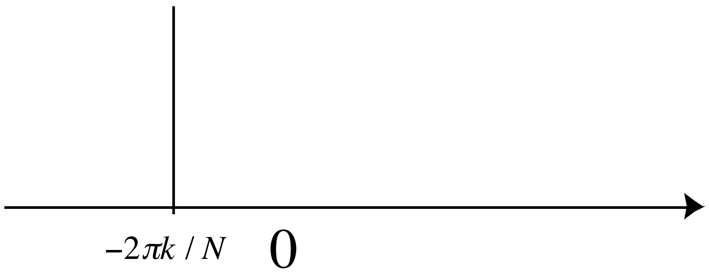
$x[n]$ periodic

DFT $$X[k]=\sum_{n=0}^{N-1}\underbrace{x[n]e^{-j(2\pi k/N)n}}_{\text{move } X_k \text{ to DC}}$$

original spectrum



$*$



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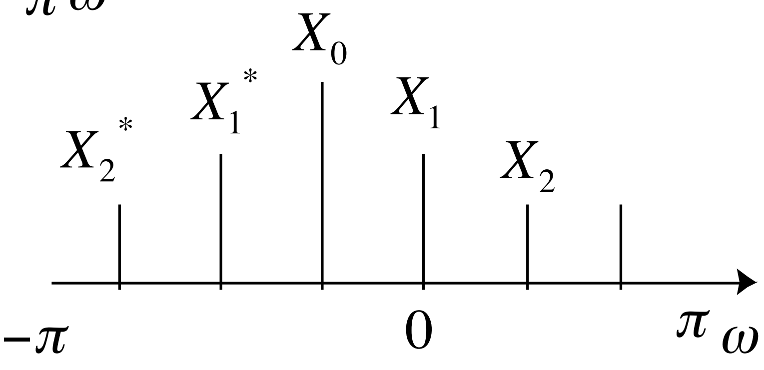
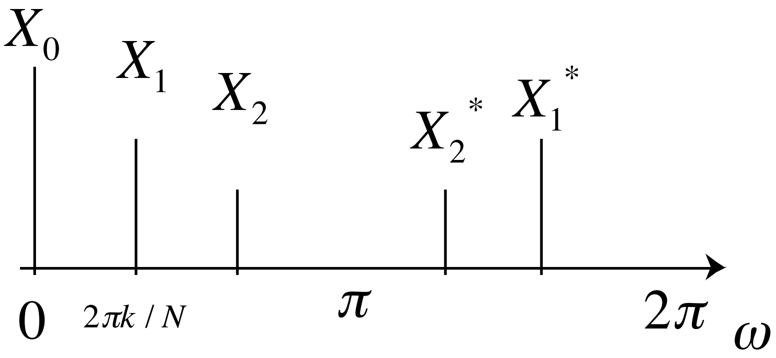


shifted spectrum

$x[n]$ periodic

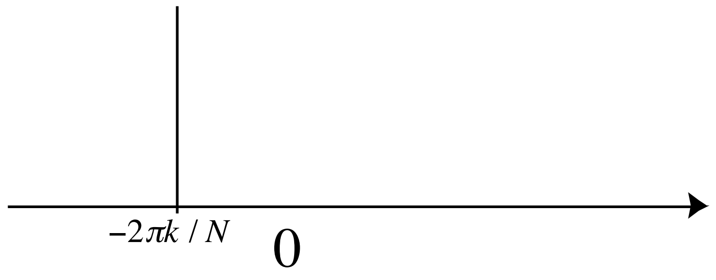
DFT $$X[k]=\sum_{n=0}^{N-1}\underbrace{x[n]e^{-j(2\pi k/N)n}}_{\text{move } X_k \text{ to DC}}$$

original spectrum

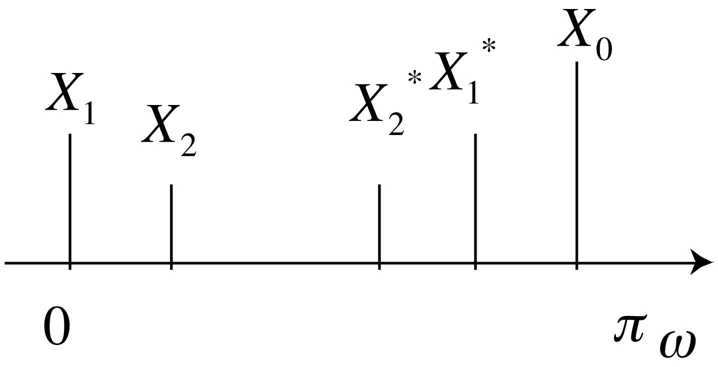


ω>π aliases of negative frequency components

$*$



$=$



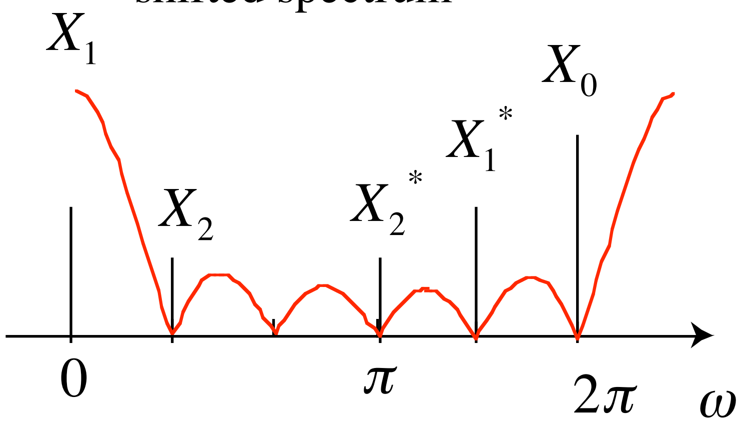
shifted spectrum

$x[n]$ periodic

DFT $$X[k] = \underbrace{\sum_{n=0}^{N-1}} x[n] e^{-j(2\pi k/N)n}$$

FIR low pass filter to measure DC
(N point running sum, {1,1,1…1})
zeros @ harmonics

shifted spectrum



$\Rightarrow \quad X_1$

FIR filter w/ zeros
at harmonics

$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$ Impulse response of 3pt averager

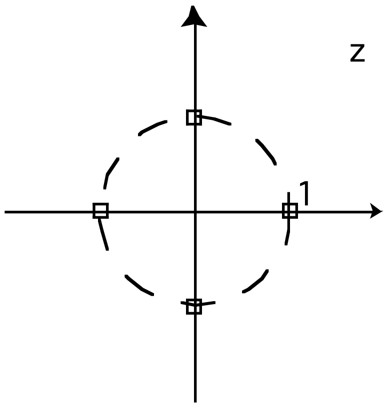
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n}$$

$$= x[0] + x[1]e^{-j(2\pi k/4)} + x[2]e^{-j(2\pi k/4)2} + x[3]e^{-j(2\pi k/4)3}$$

$$= 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

$$= 1 + e^{-j\frac{3\pi}{4}k}\left(e^{j\frac{\pi}{4}k} + e^{-j\frac{\pi}{4}k}\right)$$

$$= 1 + 2\left[-\sqrt{2}(1+j)\right]^{k} \cos\left(\tfrac{\pi}{4}k\right)$$



k=0, 1, 2, 3

$$X[k] = \{3, \quad -i, \quad 1, \quad i\}$$

$$DC \quad e^{-j(\pi/2)} \quad e^{-j(\pi)} \quad e^{-j\left(\frac{3\pi}{2}\right)}$$

check:

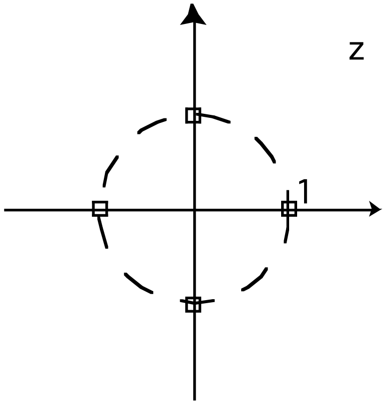
```
»x=[1 1 1 0];
»X=fft(x)
X =
   3.0000      0 - 1.0000i   1.0000      0 + 1.0000i
»fftshift(X)
ans =
   1.0000       0 + 1.0000i   3.0000      0 - 1.0000i
```

Note: 0>ω>2π

only extract limited number of frequencies due to N samples per period

$$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n}$$



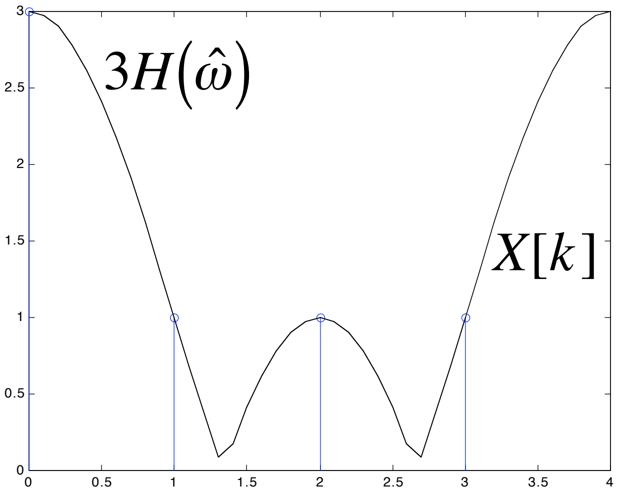
$$= 1 + 2\left[-\sqrt{2}(1+j)\right]^{k} \cos\left(\tfrac{\pi}{4} k\right)$$

k=0, 1, 2, 3

$$X[k] = \{3, \quad -i, \quad 1, \quad i\}$$

$$DC \quad e^{-j(\pi/2)} \quad e^{-j(\pi)} \quad e^{-j\left(\frac{3\pi}{2}\right)}$$

$$DC \quad e^{-j(\pi/2)} \quad e^{-j\left(\frac{-3\pi}{2}\right)} \quad e^{-j\left(-\frac{\pi}{2}\right)}$$



$$H(\hat{\omega}) = \tfrac{1}{3} e^{-j\hat{\omega}} (1 + 2\cos\hat{\omega})$$

Note: 0>ω>2π

only extract limited number of frequencies due to N samples per period

Redo like homework

$$x[n] = \{1, \quad 1, \quad 1, \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^{3} x[n] e^{-j(2\pi k/4)n} = \sum_{n=0}^{2} e^{-j(2\pi k/4)n}$$

$$= \frac{1 - e^{-j(2\pi k/4)3}}{1 - e^{-j(2\pi k/4)}} = \frac{1 - e^{-j3\pi k/2}}{1 - e^{-j\pi k/2}}$$

$$= \frac{1 - j^k}{1 - (-j)^k}$$

k≠0

Remember

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a}$$

$$X[k] = \sum_{n=0}^{2} e^{-j(2\pi 0/4)n} = \sum_{n=0}^{2} e^{-j0} = \sum_{n=0}^{2} 1 = 3$$

k=0

$$X[1] = \frac{1-j}{1+j}\frac{1-j}{1-j} = \frac{1-2j-1}{2} = -j$$

$$X[2] = \frac{1-j^2}{1-(-j)^2} = \frac{1-(-1)}{1-(-1)} = 1$$

$$X[3] = \frac{1-j^3}{1-(-j)^3} = \frac{1-(-j)}{1-j} = \frac{1+j}{1-j}\frac{1+j}{1+j} = \frac{1+2j-1}{1+2} = j$$

$$X[k] = \{3, -i, 1, i\}$$

k={0,1,2,3}

How does a Fast Fourier Transform (FFT) work?

2 Point DFT

FFT is an efficient way of calculating a DFT.

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \qquad N=2$$

$$= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/2)1}$$

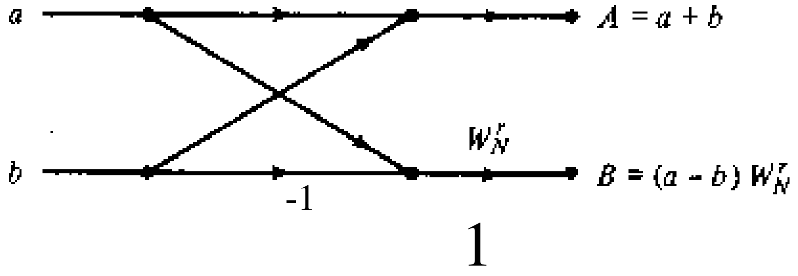
$$= x[0] + x[1]e^{-j(\pi k)}$$

$$X_2[0] = x[0] + x[1] \qquad X_2[1] = x[0] - x[1]$$

$$\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$$

coefficients block

FFT butterfly



$$W_2^k = e^{-j(2\pi k/2)}$$

$$W_2^0 = 1$$

N^2=4 mult
N^2-N=2 adds

4 Point DFT

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \qquad \text{N=4}$$

$$= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/4)1} + x[2]e^{-j(2\pi k/4)2} + x[3]e^{-j(2\pi k/4)3}$$

$$= x[0] + x[1]e^{-j\left(\frac{\pi}{2}k\right)} + x[2]e^{-j(\pi k)} + x[3]e^{-j\left(\frac{3\pi}{2}k\right)}$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$ running sum

$$X[1] = x[0] + x[1]e^{-j\left(\frac{\pi}{2}\right)} + x[2]e^{-j(\pi)} + x[3]e^{-j\left(\frac{3\pi}{2}\right)}$$

$$= x[0] - jx[1] - x[2] + jx[3]$$

$$X[2] = x[0] + x[1]e^{-j(\pi)} + x[2]e^{-j(2\pi)} + x[3]e^{-j(3\pi)}$$

$$= x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + x[1]e^{-j\left(\frac{3\pi}{2}\right)} + x[2]e^{-j(3\pi)} + x[3]e^{-j\left(\frac{9\pi}{2}\right)}$$

$$= x[0] + jx[1] - x[2] - jx[3]$$

$$X[4] = x[0] + x[1]e^{-j(2\pi)} + x[2]e^{-j(4\pi)} + x[3]e^{-j(6\pi)}$$

$$= x[0] + x[1] + x[2] + x[3]$$ alias of X[0]

N^2=16 mult
N^2-N=12 adds

4 Point DFT

DFT $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$$ N=4

$$= x[0] + x[1]e^{-j\left(\frac{\pi}{2}k\right)} + x[2]e^{-j(\pi k)} + x[3]e^{-j\left(\frac{3\pi}{2}k\right)}$$

$X[0] = x[0] + x[1] + x[2] + x[3]$ running sum

$X[1] = x[0] - jx[1] - x[2] + jx[3]$

$X[2] = x[0] - x[1] + x[2] - x[3]$

$X[3] = x[0] + jx[1] - x[2] - jx[3]$

$X[4] = x[0] + x[1] + x[2] + x[3]$ alias of X[0]

$X[5] = x[0] - jx[1] - x[2] + jx[3]$ alias of X[1]

periodic in frequency

only extract limited number of frequencies due to N samples per period

DFT

N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3] \qquad 1 \;\; 1 \;\; 1 \;\; 1$$

$$X_4[1] = x[0] - jx[1] - x[2] + jx[3] \qquad 1 \;\; -j \;\; -1 \;\; j$$

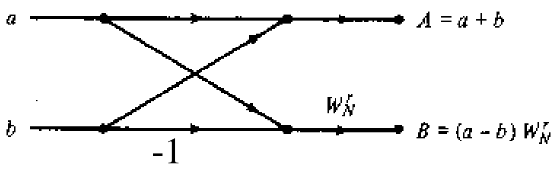
$$X_4[2] = x[0] - x[1] + x[2] - x[3] \qquad 1 \;\; -1 \;\; 1 \;\; -1$$

$$X_4[3] = x[0] + jx[1] - x[2] - jx[3] \qquad 1 \;\; j \;\; -1 \;\; -j$$

N=2

$$X_2[0] = x[0] + x[1] \qquad 1 \;\; 1$$

$$X_2[1] = x[0] - x[1] \qquad 1 \;\; -1$$



even: X[0],X[2]

$$\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{array}$$

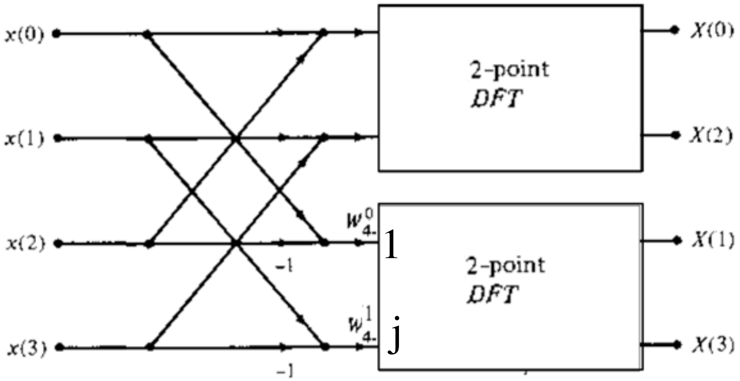
$$X_4[0] = \left[x[0] + x[2]\right] + \left[x[1] + x[3]\right] = X_2[0]_{even}$$

$$X_4[2] = \left[x[0] + x[2]\right] - \left[x[1] + x[3]\right] = X_2[1]_{even}$$

$$W_4^k = e^{-j(2\pi k/4)}$$

Looks like a 2pt FFT
of combined signals

recursive



odd: X[1],X[3]

$$\begin{array}{cc|cc} 1 & -j & -1 & j \\ 1 & j & -1 & -j \end{array}$$

Looks like a 2pt FFT
of combined signals

$$X_4[1] = \left[x[0] - x[2]\right] + j\left[-x[1] + x[3]\right] = X_2[0]_{odd}$$

$$X_4[3] = \left[x[0] - x[2]\right] - j\left[-x[1] + x[3]\right] = X_2[1]_{odd}$$

FFT

fewer multiplies/adds

$N/2\log_2 N$=4 mult

DFT

N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] - jx[1] - x[2] + jx[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

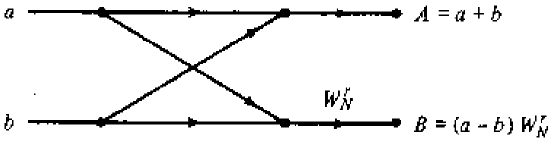
$$X[3] = x[0] + jx[1] - x[2] - jx[3]$$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{array}$$

N=2

$$X_2[0] = x[0] + x[1] \quad 1 \;\; 1$$

$$X_2[1] = x[0] - x[1] \quad 1 \;\; -1$$

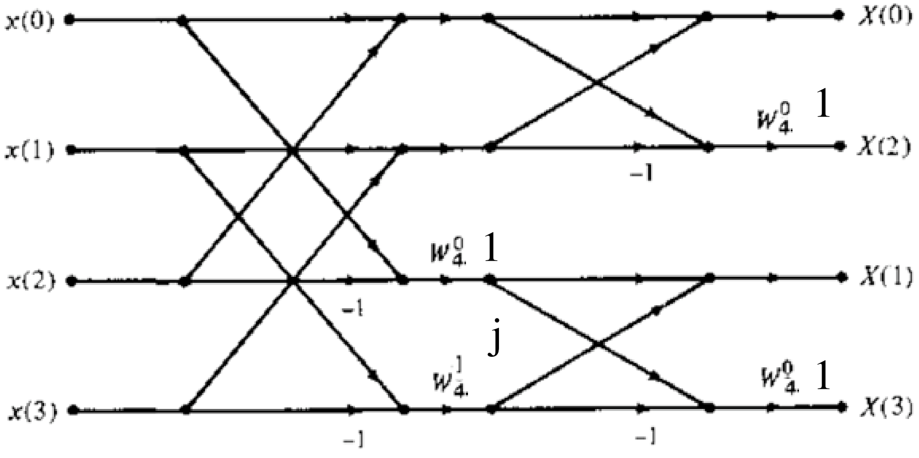


even: X[0],X[2]

$$\left.\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right| \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}$$

odd: X[1],X[3]

$$\left.\begin{array}{cc} 1 & -j \\ 1 & j \end{array}\right| \begin{array}{cc} -1 & j \\ -1 & -j \end{array}$$



4pt FFT butterfly

$N/2\log_2 N$=4 mult

$N\log_2 N$=8 add

DFT $$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi k/N)n}$$ N=8

$$=x[0]e^{-j(2\pi k/4)0}+x[1]e^{-j(2\pi k/8)1}+x[2]e^{-j(2\pi k/8)2}+x[3]e^{-j(2\pi k/8)3}$$

$$x[4]e^{-j(2\pi k/8)4}+x[5]e^{-j(2\pi k/8)5}+x[6]e^{-j(2\pi k/8)6}+x[7]e^{-j(2\pi k/8)7}$$

$$=x[0]+x[1]e^{-j\left(\frac{\pi}{4}k\right)}+x[2]e^{-j\left(\frac{\pi}{2}k\right)}+x[3]e^{-j\left(\frac{3\pi}{4}k\right)}$$

$$x[4]e^{-j(\pi k)}+x[5]e^{-j\left(\frac{5\pi}{4}k\right)}+x[6]e^{-j\left(\frac{6\pi}{4}k\right)}+x[7]e^{-j\left(\frac{7\pi}{4}k\right)}$$

$$X[0]=x[0]+x[1]+x[2]+x[3]+x[4]+x[5]+x[6]+x[7]$$ running sum

$$X[1]=x[0]+\tfrac{\sqrt{2}}{2}(1-j)x[1]-jx[2]-\tfrac{\sqrt{2}}{2}(1+j)x[3]$$

$$-x[4]+\tfrac{\sqrt{2}}{2}(-1+j)x[5]+jx[6]+\tfrac{\sqrt{2}}{2}(1+j)x[7]$$

$$X[2]=x[0]-jx[1]-x[2]+jx[3]+x[4]-jx[5]-x[6]+jx[7]$$

$$X[3]=x[0]+\tfrac{\sqrt{2}}{2}(-1-j)x[1]+jx[2]+\tfrac{\sqrt{2}}{2}(1-j)x[3]$$

$$-x[4]+\tfrac{\sqrt{2}}{2}(1+j)x[5]-jx[6]+\tfrac{\sqrt{2}}{2}(-1+j)x[7]$$

$$\vdots$$

N^2=64mult

N^2-N=56 add

DFT $$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\left(2\pi k/N\right)n}\qquad \text{N=8}$$

$$=x[0]+x[1]e^{-j\left(\frac{\pi}{4}k\right)}+x[2]e^{-j\left(\frac{\pi}{2}k\right)}+x[3]e^{-j\left(\frac{3\pi}{4}k\right)}$$

$$x[4]e^{-j\left(\pi k\right)}+x[5]e^{-j\left(\frac{5\pi}{4}k\right)}+x[6]e^{-j\left(\frac{6\pi}{4}k\right)}+x[7]e^{-j\left(\frac{7\pi}{4}k\right)}$$

$$\begin{array}{ccccccccc}
 & x[n] & & & & & & & \\
X[k] & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & \frac{\sqrt{2}}{2}(1-j) & \text{-j} & \frac{\sqrt{2}}{2}(-1-j) & \text{-1} & -\frac{\sqrt{2}}{2}(1-j) & \text{j} & \frac{\sqrt{2}}{2}(1+j) \\
 & 1 & \text{-j} & \text{-1} & \text{j} & 1 & \text{-j} & \text{-1} & \text{j} \\
 & 1 & \frac{\sqrt{2}}{2}(-1-j) & \text{j} & \frac{\sqrt{2}}{2}(1-j) & \text{-1} & -\frac{\sqrt{2}}{2}(-1-j) & \text{-j} & \frac{\sqrt{2}}{2}(-1+j) \\
 & 1 & \text{-1} & 1 & \text{-1} & 1 & \text{-1} & 1 & \text{-1} \\
 & 1 & \frac{\sqrt{2}}{2}(-1+j) & \text{-j} & \frac{\sqrt{2}}{2}(1+j) & \text{-1} & -\frac{\sqrt{2}}{2}(-1+j) & \text{j} & \frac{\sqrt{2}}{2}(-1-j) \\
 & 1 & \text{j} & \text{-1} & \text{-j} & 1 & \text{j} & \text{-1} & \text{-j} \\
 & 1 & \frac{\sqrt{2}}{2}(1+j) & \text{j} & \frac{\sqrt{2}}{2}(-1+j) & \text{-1} & -\frac{\sqrt{2}}{2}(1+j) & \text{-j} & \frac{\sqrt{2}}{2}(1-j)
\end{array}$$

N^2=64mult

N^2-N=56 add

DFT

N=8

even: X[0],X[2],X[4],X[6]

$$\begin{array}{c|cccc|cccc}
 & x[n] & & & & & & & \\
X[k] & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & -j & -1 & j & 1 & -j & -1 & j \\
 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
 & 1 & j & -1 & -j & 1 & j & -1 & -j
\end{array}$$

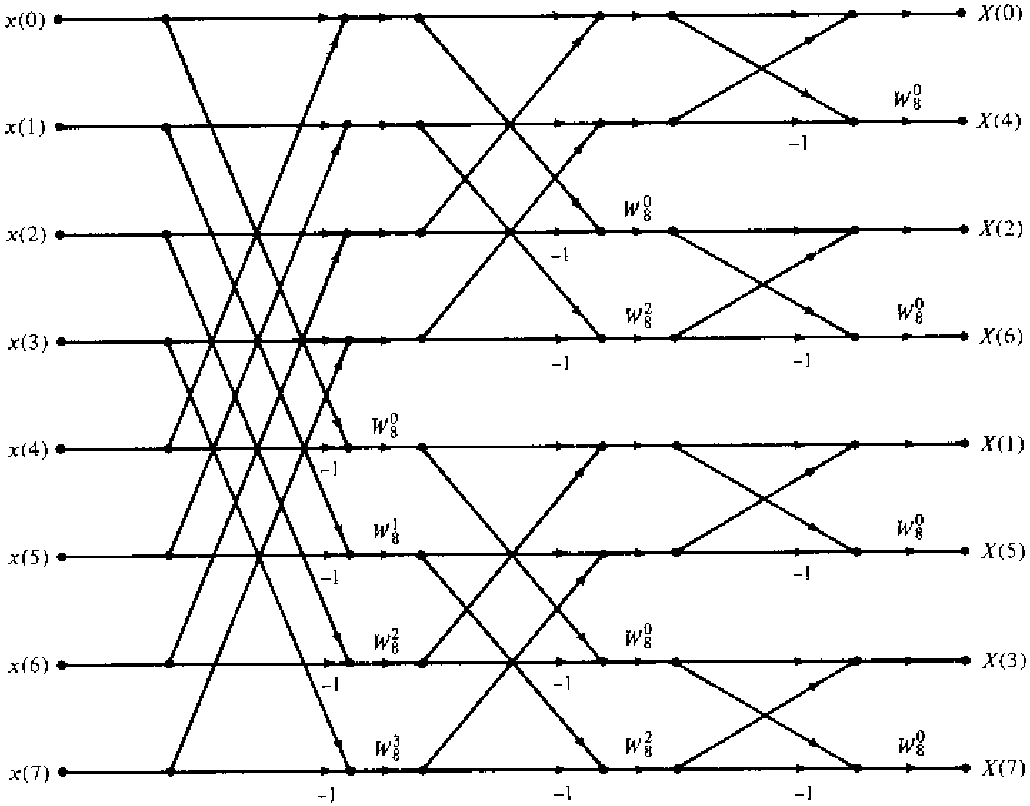
N=4

$$\boxed{\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{array}}$$

odd: X[1],X[3],X[5],X[7]

$$\begin{array}{cccc|cccc}
1 & \frac{\sqrt{2}}{2}(1-j) & -j & \frac{\sqrt{2}}{2}(-1-j) & -1 & -\frac{\sqrt{2}}{2}(1-j) & j & \frac{\sqrt{2}}{2}(1+j) \\
1 & \frac{\sqrt{2}}{2}(-1-j) & j & \frac{\sqrt{2}}{2}(1-j) & -1 & -\frac{\sqrt{2}}{2}(-1-j) & -j & \frac{\sqrt{2}}{2}(-1+j) \\
1 & \frac{\sqrt{2}}{2}(-1+j) & -j & \frac{\sqrt{2}}{2}(1+j) & -1 & -\frac{\sqrt{2}}{2}(-1+j) & j & \frac{\sqrt{2}}{2}(-1-j) \\
1 & \frac{\sqrt{2}}{2}(1+j) & j & \frac{\sqrt{2}}{2}(-1+j) & -1 & -\frac{\sqrt{2}}{2}(1+j) & -j & \frac{\sqrt{2}}{2}(1-j)
\end{array}$$

8pt FFT



$$W_8^k = e^{-j(2\pi k/8)}$$

$N/2\log_2 N=12$ mult
$N\log_2 N=24$ adds

$$x(t) = t \quad 0 \le t < T_0$$

$$X_0 = \frac{T_0}{2} \qquad X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

$$x[n] = x(nT_s) = nT_s \quad 0 \le n < \frac{T_0}{T_s}$$

$$T_s = 0 \qquad T_0 = NT_s$$

$$x[n] = x(n) = n \quad 0 \le n < \frac{NT_s}{T_s}$$

$$x(t) = \frac{N}{2} + \sum_{k=1}^{N-1} \frac{N}{\pi k} \cos\left(2\pi k \frac{1}{N} n + \tfrac{\pi}{2}\right)$$

FFT Convolution

$$y[n] = h[n] * x[n] \Leftrightarrow Y(\hat{\omega}) = H(\hat{\omega})X(\hat{\omega})$$

sample domain

frequency domain

$H[k]$ sampled version of $H(\hat{\omega})$

Use DFT/FFT to compute H[k] and X[k]

Y[k]=H[k]X[k]

Use IDFT/IFFT to compute Y[k]

Problem:

H[k]X[k] must have equal lengths N to multiply

Y[k] and y[n] will also have length N

BUT

Circular Convolution

$$y[n] = h[n] \otimes x[n] \Leftrightarrow Y[k] = H[k]X[k]$$

sample
domain

frequency
domain

Y[k]=H[k]X[k]

Use IDFT/IFFT to compute Y[k]

Problem:

H[k]X[k] must have equal lengths N to multiply

Y[k] and y[n] will also have length N

BUT

Long Division

$$Y(z) = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - z^{-1}\right)}$$

$$= \frac{1}{-2z^{-2} + z^{-1} + 1}$$

$$\begin{array}{r l}
 & 1 - z^{-1} + 3z^{-2} - 5z^{-3} \\
\cline{2-2}
= 1 + z^{-1} - 2z^{-2} \big) & 1 \\
 & \underline{1 + z^{-1} - 2z^{-2}} \\
 & \quad -z^{-1} + 2z^{-2} \\
 & \quad \underline{-z^{-1} - z^{-2} + 2z^{-3}} \\
 & \qquad\qquad 3z^{-2} - 2z^{-3} \\
 & \qquad\qquad \underline{3z^{-2} + 3z^{-3} - 6z^{-4}} \\
 & \qquad\qquad\quad -5z^{-3} + 6z^{-4}
\end{array}$$

compare

$$y[n] = \frac{2}{3}\left(-2\right)^{n} u[n] + 1/3\, u[n]$$

={1,-1,3,-5}

n=0,1,2,3

$$y[n] = \delta[n] - \delta[n-1] + 3\delta[n-2] - 5\delta[n-3] + \cdots$$

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$$

N=2

N=4

$$= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/2)1}$$

$$= x[0] + x[1]e^{-j(\pi k)}$$

$$X_2[0] = x[0] + x[1]$$

$$X_2[1] = x[0] - jx[1]$$

Compare with N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$

$$X_4[2] = x[0] + x[2] - (x[1] + x[3])$$

$$X_4[1] = x[0] - x[2] - j(x[1] - x[3])$$

$$X_4[3] = x[0] - x[2] - j(x[3] - x[1])$$

$$X[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] - jx[1] - x[2] + jx[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + jx[1] - x[2] - jx[3]$$

1	1	1	1	1	1	1	1
1	$\frac{\sqrt{2}}{2}(1-j)$	-j	$\frac{\sqrt{2}}{2}(-1-j)$	-1	$-\frac{\sqrt{2}}{2}(1-j)$	j	$\frac{\sqrt{2}}{2}(1+j)$
1	-j	-1	j	1	-j	-1	j
1	$\frac{\sqrt{2}}{2}(-1-j)$	j	$\frac{\sqrt{2}}{2}(1-j)$	-1	$-\frac{\sqrt{2}}{2}(-1-j)$	-j	$\frac{\sqrt{2}}{2}(-1+j)$
1	-1	1	-1	1	-1	1	-1
1	$\frac{\sqrt{2}}{2}(-1+j)$	-j	$\frac{\sqrt{2}}{2}(1+j)$	-1	$-\frac{\sqrt{2}}{2}(-1+j)$	j	$\frac{\sqrt{2}}{2}(-1-j)$
1	j	-1	-j	1	j	-1	-j
1	$\frac{\sqrt{2}}{2}(1+j)$	j	$\frac{\sqrt{2}}{2}(-1+j)$	-1	$-\frac{\sqrt{2}}{2}(1+j)$	-j	$\frac{\sqrt{2}}{2}(1-j)$

1	1	1	1
1	-j	-1	j
1	-1	1	-1
1	j	-1	-j

1	1
1	-1
1	1
1	-1

1	1
-j	j
-1	-1
j	-j

1	1
1	-1

Even 8pt DFT

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j \\
\hline
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{array}$$

4pt DFT

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{array}$$

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{array}$$

odd

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\frac{\sqrt{2}}{2}(1-j) & \frac{\sqrt{2}}{2}(-1-j) & -\frac{\sqrt{2}}{2}(1-j) & \frac{\sqrt{2}}{2}(1+j) \\
-j & j & -j & j \\
\frac{\sqrt{2}}{2}(-1-j) & \frac{\sqrt{2}}{2}(1-j) & -\frac{\sqrt{2}}{2}(-1-j) & \frac{\sqrt{2}}{2}(-1+j) \\
\hline
-1 & -1 & -1 & -1 \\
\frac{\sqrt{2}}{2}(-1+j) & \frac{\sqrt{2}}{2}(1+j) & -\frac{\sqrt{2}}{2}(-1+j) & \frac{\sqrt{2}}{2}(-1-j) \\
j & -j & j & -j \\
\frac{\sqrt{2}}{2}(1+j) & \frac{\sqrt{2}}{2}(-1+j) & -\frac{\sqrt{2}}{2}(1+j) & \frac{\sqrt{2}}{2}(1-j)
\end{array}$$

-j

bot=-top

Long Division

$$Y(z) = \frac{1 + 1.9z^{-1}}{\left(1 + 2z^{-1}\right)\left(1 - z^{-1}\right)} \quad = \frac{1 + 1.9z^{-1}}{-2z^{-2} + z^{-1} + 1}$$

$$
\begin{array}{r l}
 & 1 + 0.9z^{-1} + 1.1z^{-2} + 0.7z^{-3} \\ \cline{2-2}
= 1 + z^{-1} - 2z^{-2} & \big)\, 1 + 1.9z^{-1} \\
 & \underline{1 + z^{-1} - 2z^{-2}} \\
 & \quad 0.9z^{-1} + 2z^{-2} \\
 & \quad \underline{0.9z^{-1} + 0.9z^{-2} - 1.8z^{-3}} \\
 & \qquad\qquad 1.1z^{-2} + 1.8z^{-3} \\
 & \qquad\qquad \underline{1.1z^{-2} + 1.1z^{-3} - 2.2z^{-4}} \\
 & \qquad\qquad\qquad 0.7z^{-3} + 2.2z^{-4}
\end{array}
$$

right sided sequence

$$y[n] = \delta[n] + 0.9\delta[n-1] + 1.1\delta[n-2] + 0.7\delta[n-3] + \cdots$$

compare

$y[n] = 0.033\left(-2\right)^{n} u[n] + 0.9u[n]$ ={1, 0.9, 1.1, 0.7} n=0,1,2,3

Long Division

$$Y(z) = \frac{1+1.9z^{-1}}{\left(1+2z^{-1}\right)\left(1-z^{-1}\right)} \quad = \frac{1+1.9z^{-1}}{-2z^{-2}+z^{-1}+1}$$

$$\begin{array}{r l}
 & -0.95z-0.975z^{2}-0.9625z^{3} \\
\hline
=-2z^{-2}+z^{-1}+1\big) & 1.9z^{-1}+1 \\
 & \underline{1.9z^{-1}-0.95-0.95z} \\
 & \quad 1.95+0.95z \\
 & \quad \underline{1.95-0.975z-0.975z^{-3}} \\
 & \qquad\quad 1.925z^{-2}+0.975z^{-3} \\
 & \qquad\quad \underline{1.925z^{-2}-0.9625z^{-3}-0.9625z^{-4}}
\end{array}$$

left sided sequence

$$y[n] = -0.95\delta[n+1]-0.975\delta[n+2]+0.9625\delta[n+3]+\cdots$$

compare

$$y[n] = 0.033\left(-2\right)^{n}u[-n-1]+0.9u[-n-1]$$ ={0.95, 0.975, 0.963}

n={-1, -2, -3}

$$x[n]=\{1,\quad 0,\quad -1,\quad 0\}$$

$$x[n]=\cos\left(\frac{2\pi}{4}n\right)$$

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j(2\pi k/N)n} = \sum_{n=0}^{3}x[n]e^{-j(2\pi k/4)n}$$

$$= x[0]+x[1]e^{-j(2\pi k/4)}+x[2]e^{-j(2\pi k/4)2}+x[3]e^{-j(2\pi k/4)3}$$

$$= x[0]+x[2]e^{-j(2\pi k/4)2}$$

$$=1-e^{-j\pi k}$$

$$=1-(-1)^{k}$$

$$\begin{array}{ccccc} & k=0, & 1, & 2, & 3 \\ X[k]= & \{0, & 2, & 0, & 2\} \\ & DC & e^{-j(\pi/2)} & e^{-j(\pi)} & e^{-j\left(\frac{3\pi}{2}\right)} \end{array}$$

check:

```
»x=[1 0 -1 0];
»X=fft(x)
X =
    0    2    0    2
```

Note: 0>ω>2π

only extract limited number of frequencies due to N samples per period