Impulse response

\[ y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{FIR filter} \]

\[ x[n] = \delta[n] = \begin{cases} 
1 & n = 0 \\
0 & \text{otherwise} 
\end{cases} \quad \text{Delta function} \]

\[ y[n] \big|_{x=\delta[n]} = h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \]

impulse response

\[ y[n] = \sum_{k=-\infty}^{\infty} h[n] x[n-k] \quad \text{convolution sum} \]

LTI: FIR, IIR
Frequency response

\[ y[n] = \sum_{k=0}^{M} h[k] x[n-k] \]

convolution

\[ x[n] = Ae^{j\phi} e^{j\hat{\omega}n} \]

Complex exponential input

\[ \hat{\omega} = \omega T_s \]

\[ y[n] = \sum_{k=0}^{M} h[k] Ae^{j\phi} e^{j(\hat{\omega}n-k)} \]

\[ = \left( \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k} \right) Ae^{j\phi} e^{j\hat{\omega}n} \]

let

\[ H(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{j\hat{\omega}k} \]

\[ H(\hat{\omega}) \quad \text{frequency response} \]
\[ y[n] = \sum_{k=0}^{M} h[k] x[n-k] \]  
convolution

\[ x[n] = Ae^{j\phi} e^{j\omega n} \]  
complex exponential input

\[ y[n] = H(\omega) Ae^{j\phi} e^{j\omega n} \]  
\[ H(\omega) = \sum_{k=0}^{M} h[k] e^{j\omega k} \]  
frequency response  
complex

\[ y[n] = |H(\omega)| Ae^{j(\phi + \angle H(\omega))} e^{j\omega n} \]  
output same frequency as input, but amplitude scaled and a phase shift

LTI: FIR & IIR
\[
\begin{pmatrix}
n & x[n] \\
0 & 1 \\
1 & 0 \\
2 & 0 \\
3 & 0 \\
4 & 0 \\
> 5 & 0 \\
\end{pmatrix}
\quad x[n] = \delta[n]
\]

system

\[
h[n] = y[n]_{x[n]=\delta[n]}
\]

\[
\begin{pmatrix}
1/3 \\
0 \\
1/3 \\
0 \\
1/3 \\
0 \\
\end{pmatrix}
\]

\[
y[n] = ?
\]
Ex. \[ h[n] = \frac{1}{3} \delta[n] + \frac{1}{3} \delta[n - 2] + \frac{1}{3} \delta[n - 4] \]  
FIR 
\[ y[n] = \frac{1}{3} x[n] + \frac{1}{3} x[n - 2] + \frac{1}{3} x[n - 4] \]

\[ H(\hat{\omega}) = \sum_{k=0}^{4} h[k] e^{-j\hat{\omega}k} \]

\[ = h[0] e^{-j\hat{\omega}0} + h[2] e^{-j\hat{\omega}2} + h[4] e^{-j\hat{\omega}4} \]

\[ = \frac{1}{3} + \frac{1}{3} e^{-j\hat{\omega}2} + \frac{1}{3} e^{-j\hat{\omega}4} \]

\[ = \frac{1}{3} \left( 1 + e^{-j\hat{\omega}2} + e^{-j2\hat{\omega}4} \right) \]

\[ = \frac{1}{3} e^{-j\hat{\omega}2} \left( e^{j\hat{\omega}2} + 1 + e^{-j\hat{\omega}2} \right) \]

\[ = \frac{1}{3} e^{-j\hat{\omega}2} \left( 1 + 2 \cos 2\hat{\omega} \right) \]

Also try by inspection if \( b_k \)'s symmetric, then factor out \( e^{-j\hat{\omega}(M/2)} \) where \( M \) is the order of the filter. This leaves complex conjugate paired exponentials to transform into trigonometric functions (cosines/sines).
$$H(\hat{\omega}) = \frac{1}{3} e^{-j\hat{\omega}^2} (1 + 2 \cos 2\hat{\omega})$$

$$|H(\hat{\omega})| = \frac{1}{3} |(1 + 2 \cos 2\hat{\omega})|$$

Note:  
$$|H\left(\frac{\pi}{3}\right)| = 0$$  
$$|H\left(\frac{2\pi}{3}\right)| = 0$$

$$\angle H(\hat{\omega}) = -2\hat{\omega} \quad \text{linear phase}$$  
$$\angle H(-\hat{\omega}) = -\angle H(\hat{\omega})$$

principal value of phase fn  
$$-\pi < \angle H(\hat{\omega}) < \pi \quad \text{if not, add multiples of } 2\pi$$

Want positive magnitudes,  
$$|H(\hat{\omega})| \geq 0$$  
so absorb negative sign into phase by adding an additional  \( \pi \) at each zero
\[ H(\hat{\omega}) = \frac{1}{3} e^{-j2\hat{\omega}} (1 + 2\cos 2\hat{\omega}) \]

\[ |H(\hat{\omega})| = \frac{1}{3} |(1 + 2\cos 2\hat{\omega})| \]

Note: \[ |H\left(\frac{\pi}{3}\right)| = 0 \]
\[ |H\left(\frac{2\pi}{3}\right)| = 0 \]

\[ \angle H(\hat{\omega}) = -2\hat{\omega} \]
linear phase
\[ = \begin{cases} 
-2\hat{\omega} & 0 \leq \hat{\omega} < \pi/3 \\
-2\hat{\omega} + \pi & \pi/3 \leq \hat{\omega} < 2\pi/3 \\
-2\hat{\omega} + 2\pi & 2\pi/3 \leq \hat{\omega} < \pi 
\end{cases} \]

phase odd function
\[ \angle H(-\hat{\omega}) = -\angle H(\hat{\omega}) \]
principal value of phase fn
\[ -\pi < \angle H(\hat{\omega}) < \pi \]
Linear Phase

delay of \( n_0 \) sample periods

\[
y[n] = x[n - n_0]
\]

\[
H(\hat{\omega}) = e^{-jn_0}
\]

\[
|H(\hat{\omega})| = 1 \quad \angle H(\hat{\omega}) = -n_0\hat{\omega} \quad \text{linear phase}
\]

FIR filters are linear phase if the coefficients are symmetric.

higher frequencies need larger phase shifts than lower frequencies to achieve same time delay

\[
\angle H(\hat{\omega}) = -2\hat{\omega}
\]

\[
= \begin{cases} 
-2\hat{\omega} & 0 \leq \hat{\omega} < \pi/2 \\
-2\hat{\omega} + 2\pi & \pi/2 \leq \hat{\omega} < 3\pi/2 
\end{cases}
\]

Linear Phase, 2 sample delay

\[
y = \sin(\omega(t + nT_s)) 
= \sin(\omega t + \omega nT_s)
= \sin(\omega t + \phi)
\]

\[
\phi = T_s n \omega
\]
These are the pulse responses of each of the filters. The pulse response is nothing more than a positive going step response followed by a negative going step response. The pulse response is used here because it displays what happens to both the rising and falling edges in a signal.

Here is the important part: zero and linear phase filters have left and right edges that look the same, while nonlinear phase filters have left and right edges that look different.

Many applications cannot tolerate the left and right edges looking different. One example is the display of an oscilloscope, where this difference could be misinterpreted as a feature of the signal being measured. Another example is in video processing. Can you imagine turning on your TV to find the left ear of your favorite actor looking different from his right ear?

http://www.dsptool.com/ch19/4.htm

"It turns out that, within very generous tolerances, humans are insensitive to [audio] phase shifts. …" – Floyd E. Toole, PhD
Vice President Acoustical Engineering
Harman International Industries, Inc.

"For data transmission, a nonlinear phase delay causes intersymbol interference which increases error rate, particularly if the signal-to-noise ration is poor" – Digital Signal Processing in Communication Systems By Marvin E. Frerking
FREQZ Z-transform digital filter frequency response.

When N is an integer, \([H,W] = \text{FREQZ}(B,A,N)\) returns the N-point frequency vector \(W\) in radians and the N-point complex frequency response vector \(H\) of the filter \(B/A:\)

\[
H(e) = \frac{-1}{-1} b(1) + b(2)e^{-j\omega} + \ldots + b(nb+1)e^{-j\omega nb} \\
A(e) = \frac{-1}{-1} 1 + a(2)e^{-j\omega} + \ldots + a(na+1)e^{-j\omega na}
\]

given numerator and denominator coefficients in vectors \(B\) and \(A\).

\(\text{FREQZ}(B,A,...)\) with no output arguments plots the magnitude and unwrapped phase of \(B/A\) in the current figure window.

\[
H(\hat{\omega}) = \frac{1}{3} e^{-2j\hat{\omega}} (1 + 2 \cos 2\hat{\omega}) = \frac{1}{3} + \frac{2}{3} e^{-j\hat{\omega}^2} + \frac{1}{3} e^{-j\hat{\omega}^4}
\]

\[
z^{-1} = e^{-j\hat{\omega}}
\]

\[
H(\hat{\omega}) = \frac{\frac{1}{3} + \frac{2}{3} z^{-2} + \frac{1}{3} z^{-4}}{1} \\
\rightarrow \quad b(1) = \frac{1}{3}, b(2) = 0, b(3) = \frac{1}{3}, b(4) = 0, b(5) = \frac{1}{3} \\
a(1) = 1
\]

\[
\gg \text{freqz}([1/3,0,1/3,0,1/3],[1])
\]
\[
H(\hat{\omega}) = \frac{1}{3} e^{-j2\hat{\omega}} (1 + 2 \cos 2\hat{\omega}) = \frac{1}{3} + \frac{1}{3} e^{-j\hat{\omega}^2} + \frac{1}{3} e^{-j\hat{\omega}^4}
\]

>> freqz([1/3,0,1/3,0,1/3],[1])

Magnitude response is plotted on a logarithmic scale.

decibels (dB) = 20\log_{10}(|H|)

Note: In this plot the normalized frequency goes from DC to Nyquist ($\hat{\omega} = \pi$), so this is just one side. We normally plot from $-\pi < \hat{\omega} < \pi$.

Remember:
For real filter coefficients, magnitude is an even function; phase is an odd function.
\[ H(\hat{\omega}) = \frac{1}{3} e^{-j2\hat{\omega}} (1 + 2 \cos 2\hat{\omega}) = \frac{1}{3} + \frac{1}{3} e^{-j\hat{\omega}^2} + \frac{1}{3} e^{-j\hat{\omega}^4} \]

```matlab
>>[H,W]=freqz([1/3,0,1/3,0,1/3],[1]);
>>subplot(2,1,1),plot([-W,W],[abs(H),abs(H)]);
>>ylabel('phase');
>>subplot(2,1,2),plot([-W,W],[-angle(H),angle(H)]);
>>ylabel('phase');xlabel('freq');axis([-pi pi -pi pi])
```
Superposition and the frequency response

\[ x[n] = 3 + 3\cos(0.6\pi n) \]

input

\[ y[n] = \frac{1}{3} x[n] + \frac{1}{3} x[n-2] + \frac{1}{3} x[n-4] \]

FIR filter

sample domain

\[ y[n] = 1 + \cos(0.6\pi n) + 1 + \cos(0.6\pi(n - 2)) + 1 + \cos(0.6\pi(n - 4)) \]

\[ = 3 + \cos(0.6\pi n) \]

\[ + \cos(0.6\pi n)\cos(1.2\pi) + \sin(0.6\pi n)\sin(1.2\pi) \]

\[ + \cos(0.6\pi n)\cos(2.4\pi) + \sin(0.6\pi n)\sin(2.4\pi) \]

\[ = 3 + [1 + \cos(1.2\pi) + \cos(2.4\pi)]\cos(0.6\pi n) \]

\[ + [\sin(1.2\pi) + \sin(2.4\pi)]\sin(0.6\pi n) \]

\[ = 3 + A\cos(0.6\pi n) + B\sin(0.6\pi n) \]

\[ = 3 + \sqrt{A^2 + B^2} \cos(0.6\pi n + \tan^{-1}(B/A)) \]

\[ = 3 + 0.618 \cos(0.6\pi n - 0.2\pi) \]
frequency domain

\[ x[n] = 3 + 3\cos(0.6\pi n) \]

\[ |H(\hat{\omega})| = \frac{1}{3} (1 + 2 \cos 2\hat{\omega}) \quad \hat{\omega} \quad |H(0)| = 1 \quad |H(0.6\pi)| = 0.206 \]

\[ \angle H(\hat{\omega}) = -2\hat{\omega} + \pi \quad \angle H(0) = 0 \quad \angle H(-2 \cdot 0.6\pi + \pi) = -0.2\pi \]

\[ y[n] = 3(1) + 3(0.206)\cos(0.6\pi n - 0.2\pi) \]

\[ = 3 + 0.618 \cos(0.6\pi n - 0.2\pi) \]
$$|H(\hat{\omega})|$$

$$\angle H(\hat{\omega})$$

$$x = \sin(2\pi (0.5 \cdot (f_2-f_1)/T \cdot t^2 + f_1 \cdot t))$$

$$y = \text{conv}(x, h)$$

$$\text{soundsc}(y)$$
Frequency response

\[ y[n] = \sum_{k=0}^{M} h[k] x[n - k] \quad \text{convolution} \]

\[ x[n] = Ae^{j\phi} e^{j\hat{\omega}n} \quad \text{complex exponential input} \]

\[ y[n] = \sum_{k=0}^{M} h[k] Ae^{j\phi} e^{j(\hat{\omega}n - k)} \]

\[ = \left( \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k} \right) Ae^{j\phi} e^{j\hat{\omega}n} \]

\[ = H(\hat{\omega}) Ae^{j\phi} e^{j\hat{\omega}n} \]

\[ H(\hat{\omega}) \quad \text{frequency response} \]
System function

\[
y[n] = \sum_{k=0}^{M} h[k] x[n-k]
\]

convolution

\[
x[n] = z^n
\]

power sequence

of complex numbers

\[
y[n] = \sum_{k=0}^{M} h[k] z^{n-k}
\]

\[
= \left( \sum_{k=0}^{M} h[k] z^{-k} \right) z^n
\]

\[
= H(z) z^n
\]

\[
= |H(z)| e^{j \angle H(z)} z^n
\]

scaled and shifted
power sequence

\[
H(z) = \sum_{k=0}^{M} h[k] z^{-k}
\]

system function

LTI
characteristic functions of LTI systems

\[ z = e^{j \frac{2\pi}{16}} \]

sinusoid

\[ z = 0.9e^{j \frac{2\pi}{16}} \]

damped sinusoid

\[ z = 0.8 \]

exponentials
$z$-plane and sample responses $z^n$
$z = 1$

DC, $\omega = 0$

unit circle

$z = e^{\pm j\omega}$

sinusoids

$z = -1$

Nyquist sampled sinusoid
Damped sinusoids

\( z = 0.5e^{\pm j\frac{\pi}{4}} \)

\( z = 0.5e^{\pm j\frac{\pi}{2}} \)

\( z = 0.5e^{\pm j\frac{3\pi}{4}} \)

\( z = -0.5 \)

Nyquist sampled damped sinusoid

|z|<1

\( z^n \)
$z$-plane

- $z = 0$
- $z = 0.4$
- $z = 0.6$
- $z = -0.5$
- $z = -1$

$z^n$

- $z = 1$

$\omega_s/2$

2 samples/cycle

Nyquist sampled signals

$+\text{Re}(z)$ axis exponential decay

$-\text{Re}(z)$ axis
$z$-plane

$|z| > 1$ unstable

$z = \pm j0.4$

$z = \pm j0.5$

$z = \pm j1.1$

$|z| < 1$ stable

$z^n$

$z = \pm 1j$

Im$|z|$ axis

every other sample 0

$|z| > 1$ unstable

$|z| < 1$ stable
Ex. 
\[ y[n] = \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n-2] \]  
FIR

\[ h[n] = \frac{1}{3} \delta[n] + \frac{1}{3} \delta[n-1] + \frac{1}{3} \delta[n-2] \]

\[ H(z) = \sum_{k=0}^{2} h[k]z^{-k} \]

\[ = h[0]z^0 + h[1]z^{-1} + h[2]z^{-2} \]

\[ = \frac{1}{3} + \frac{1}{3} z^{-1} + \frac{1}{3} z^{-2} \]

\[ = \frac{1}{3} (z^{-1})^0 + \frac{1}{3} (z^{-1})^1 + \frac{1}{3} (z^{-1})^2 \quad \text{polynomial in } z^{-1} \]

\[ h[n] = \sum_{k=0}^{2} b_k \delta[n-k] \iff H(z) = \sum_{k=0}^{2} b_k z^{-k} \]

sequence  \iff  polynomial

n-domain (sample space)  \iff  z-domain (complex freq space)
Ex. \[ H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2} \]

\[ y[n] = H(z)z^n \]

\[ \frac{y[n]}{x[n]} = 0 \quad H(z) = 0 \quad z^2 + z + 1 = 0 \]
\[ z = \frac{1}{2}(-1 \pm j\sqrt{3}) = e^{\pm j\pi/6} \quad \text{zeros} \]

\[ \frac{y[n]}{x[n]} = \infty \quad H(z) = \infty \quad z^2 = 0 \quad \text{poles} \]

\[ z = 0 \iff x[n] = \{1, 0, 0\} \]
\[ y[n] = h[n] = \{1/3, 1/3, 1/3\} \]

\[ |H(z)| = \frac{y[n]}{x[n]} \]

But for \( n = 1,2 \)

\[ |H(z)| = \frac{1/3}{(0)} = \infty \]
system response $|H(z)|$

pz plot

frequency response

$|H(\omega)| = |H(z)|_{z = e^{j\omega}}$
\[ h[n] \Leftrightarrow H(z) = \sum_{k=0}^{M} b_k z^{-k} \]

impulse response sequence

\[ 1 - \sum_{k=1}^{N} a_k z^{-k} \]

system function polynomial

\[ X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} \]

sequence \( \Leftrightarrow \) polynomial

\( n \)-domain (sample space) \( \Leftrightarrow \) z-domain (complex freq space)

\( H(z) \) is the z-transform of the impulse response \( h[n] \). LTI
z-transform

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \iff X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} \]

sequence \iff polynomial

n-domain (sample space) \iff z-domain (complex freq space)

Why’re we interested?

\[ y[n] = \sum_{k=0}^{M} h[k] x[n - k] \quad \text{convolution sum} \]

\[ Y(z) = H(z) X(z) \quad \text{polynomial multiplication} \]
z-transform

\[ x[n] \iff X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} \]

Impulse

\[ x[n] = \delta[n] \iff X(z) = \sum_{k=-\infty}^{\infty} \delta[k]z^{-k} = z^{-0} = 1 \]

Ex.

\[ y[n] = h[n] \ast x[n] = h[n] \ast \delta[n] \updownarrow z \]
\[ Y(z) = H(z) \cdot 1 = H(z) \updownarrow z \]
\[ y[n] = h[n] \]

H(z) is the z-transform of the impulse response h[n]

LTI