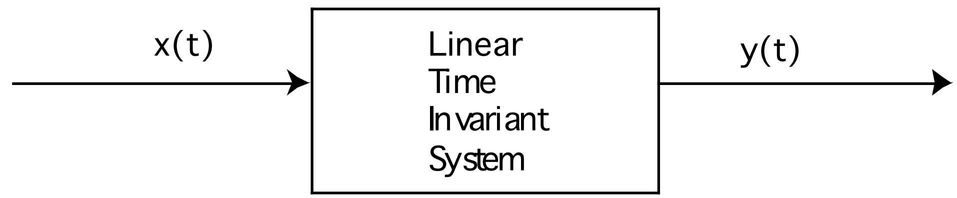
Linear Time Invariant Systems



Linearity

input $\quad\quad\quad\quad$ output

$$Ax_1(t)+Bx_2(t) \longrightarrow Ay_1(t)+By_2(t)$$

scaling & superposition

Time invariance

$$x(t-\tau) \longrightarrow y(t-\tau)$$

Characteristic Functions

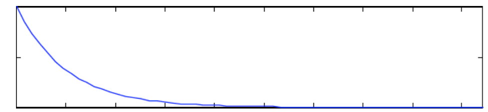
$$e^{st} \qquad s=a+jb$$

complex exponentials

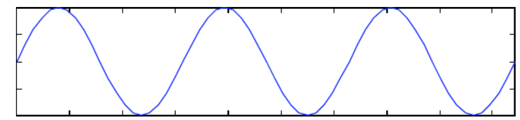
Complex Exponential Signals

$s = a + jb$ $\qquad e^{st}$

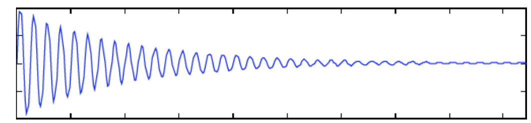
$s = \sigma$ $\qquad e^{-\sigma t}$ $\qquad$ exponential decay



$s = \pm j\omega$ $\qquad e^{\pm j\omega t}$ $\qquad$ sinusoids



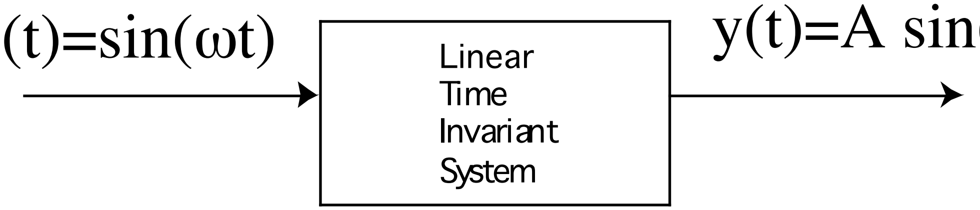
$s = -\sigma \pm j\omega$ $\qquad e^{-\sigma \pm j\omega t}$ $\qquad$ exponential sinusoids

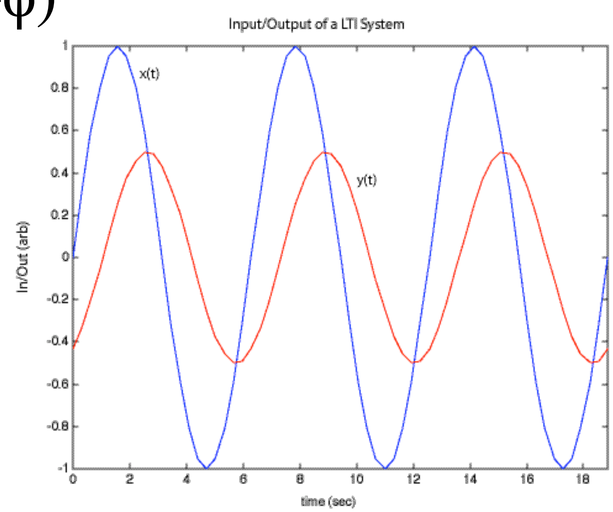


$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

characteristic functions of LTI systems

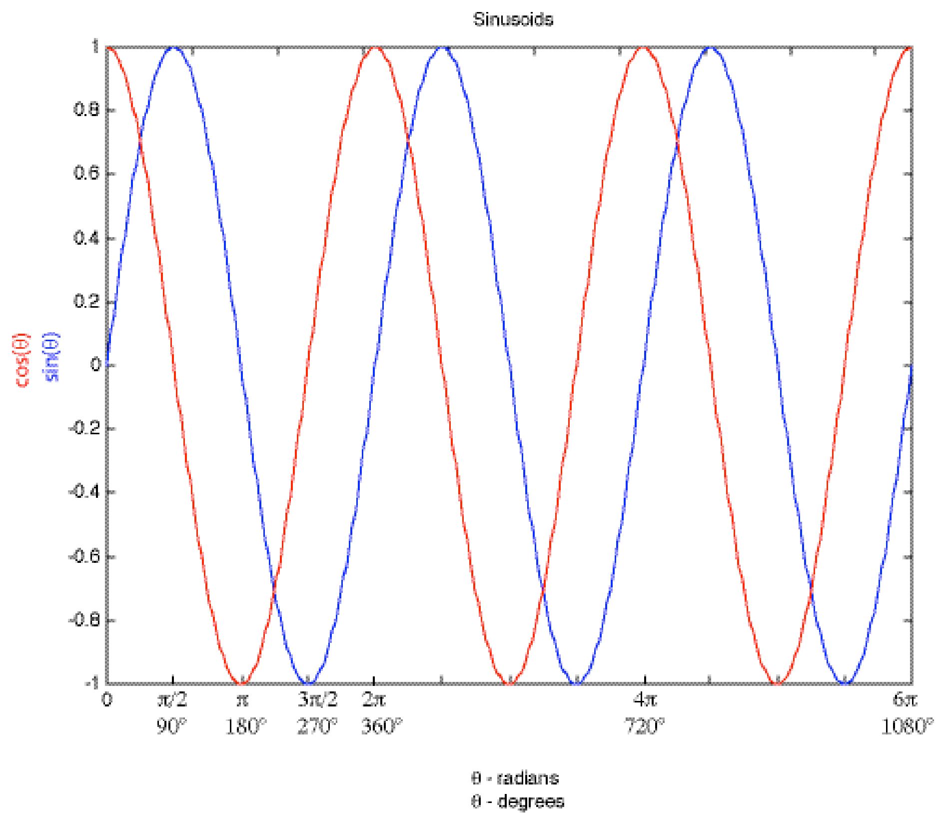
$x(t) = \sin(\omega t)$ $\qquad y(t) = A\sin(\omega t + \phi)$





output has same frequency as input
but is scaled and phase shifted

Sinusoids



θ(rad)	y_1=sin(θ)	y_2=cos(θ)
0	0	1
π/6	.5	0.866
π /4	0.707	0.707
π /3	0.866	0.5
π /2	1	0
π	0	–1
3π/2	–1	0
2π	0	1

Periodic

$$y(\theta)=y(\theta+2\pi n)$$

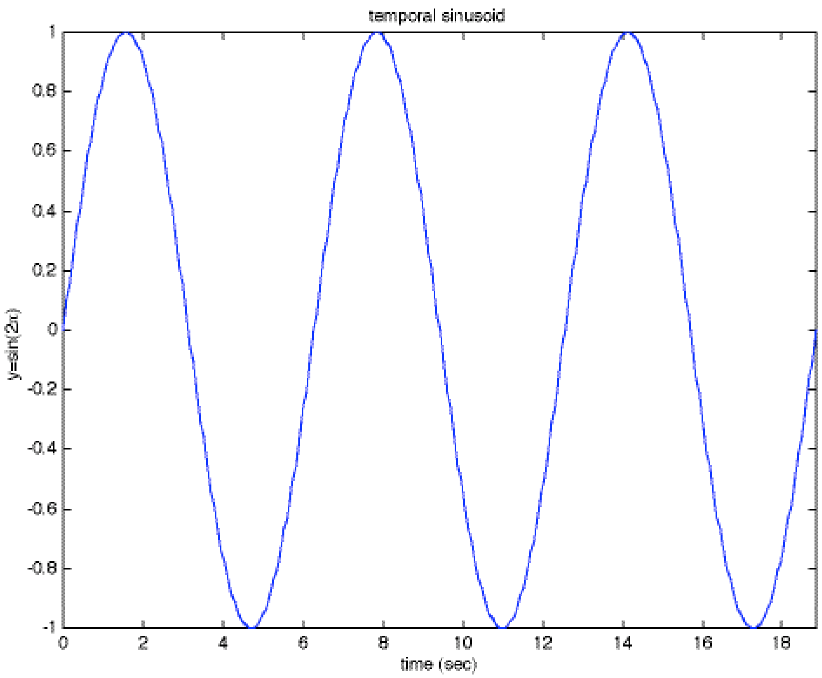
sine odd

$$\sin(-\theta)=\sin(\theta)$$

cosine even

$$\cos(-\theta)=-\cos(\theta)$$

Continuous sinusoids $\theta=\theta(t)$



$y(t)=A\sin(\omega t+\phi)$

$y(t)=A\sin(2\pi ft+\phi)$

Parameters:

A: amplitude

ϕ: phase (radians)

ω: radian frequency (radians/sec)

or

f: frequency (cycles/sec-Hz)

Relations:

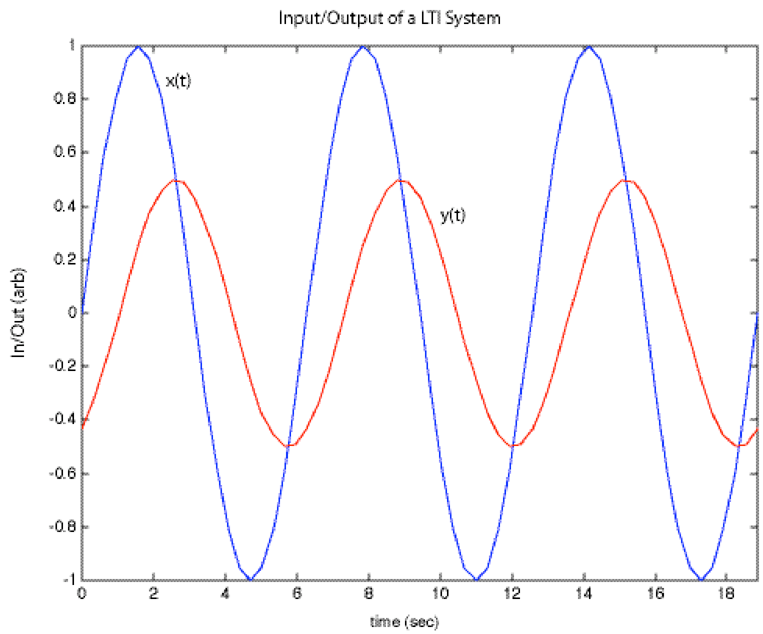
$\omega=2\pi f$ rad/sec= (2π rad/cycle)*cycle/sec

T:period (sec/cycle) $y(t)=y(t+T)$

$T = 1/f= 2\pi/\omega$

sec/cycle= 1/ (cycle/sec)= (2π rad/cycle)/(rad/sec)

Continuous sinusoids   θ=θ(t)



$y(t)=A \sin(\omega t+\phi)$
$y(t)=A \sin(2\pi f t+\phi)$

Parameters:

A: amplitude

φ: phase (radians)

ω: radian frequency (radians/sec)

or

f: frequency (cycles/sec-Hz)

Phase shift

In: $x(t)=1\sin(t)$
Out: $y(t)=0.5\sin(t-\pi/3)$

$x(0)=0$

$y(\pi/3)=0$

$(t-\pi/3)$
delay of $\pi/3$
plot moves to the right

Sampled Continuous Sinusoid

Continuous Sinusoid

$$y(t) = \sin(2\pi t)$$

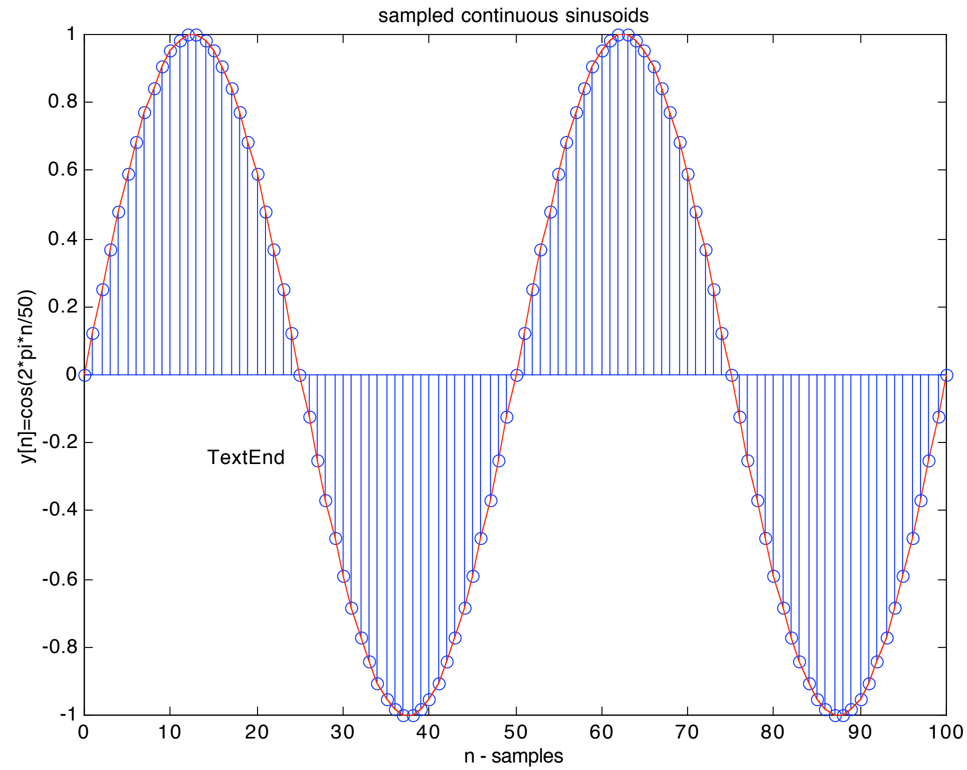
Sample rate: $T_s = \frac{1}{50}\sec$

$$t = nT_s$$

Discrete Sinusoid

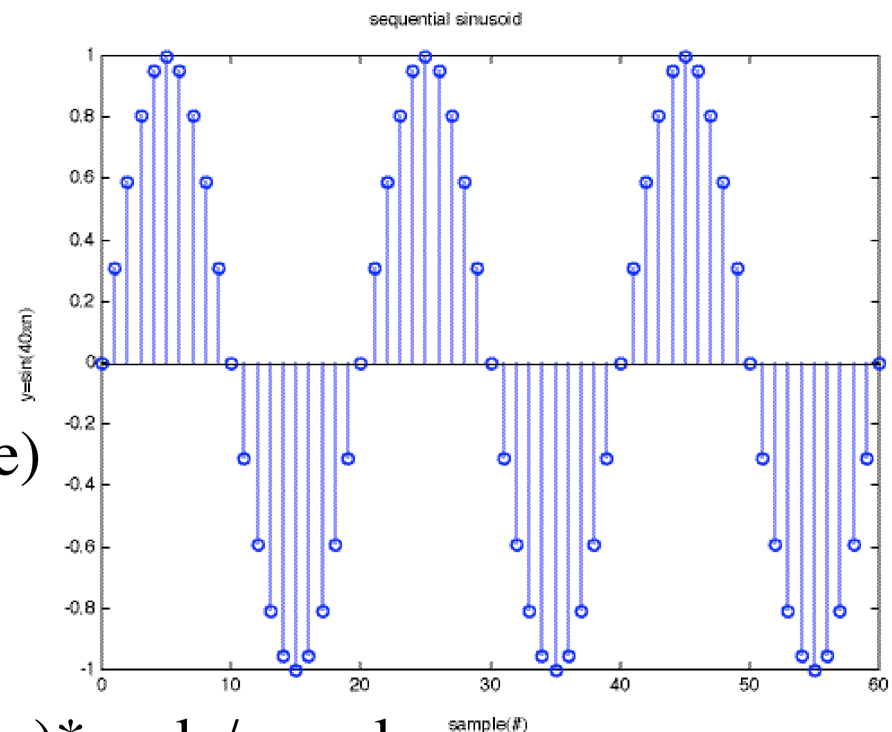
$$y[n] = \sin(2\pi \cdot \frac{1}{50} \cdot n)$$

$$y[n] = \sin(\frac{\pi}{25} \cdot n)$$



n	y[n]
0	0
1	0.125
2	0.249
3	0.368
4	0.4818

Discrete sinusoids $\theta=\theta[n]$ $n=0, 1, 2\ldots$



$y[n]=A \sin(\omega n+\phi)$

$y[n]=A \sin(2\pi f n+\phi)$

A: amplitude
ϕ: phase (radians)
ω: radian frequency (radians/sample)
f: frequency (cycles/sample)

Relations:

$\omega=2\pi f$ rad/sample= (2π rad/cycle)*cycle/sample

N:period (samples/repeating cycle [integer])

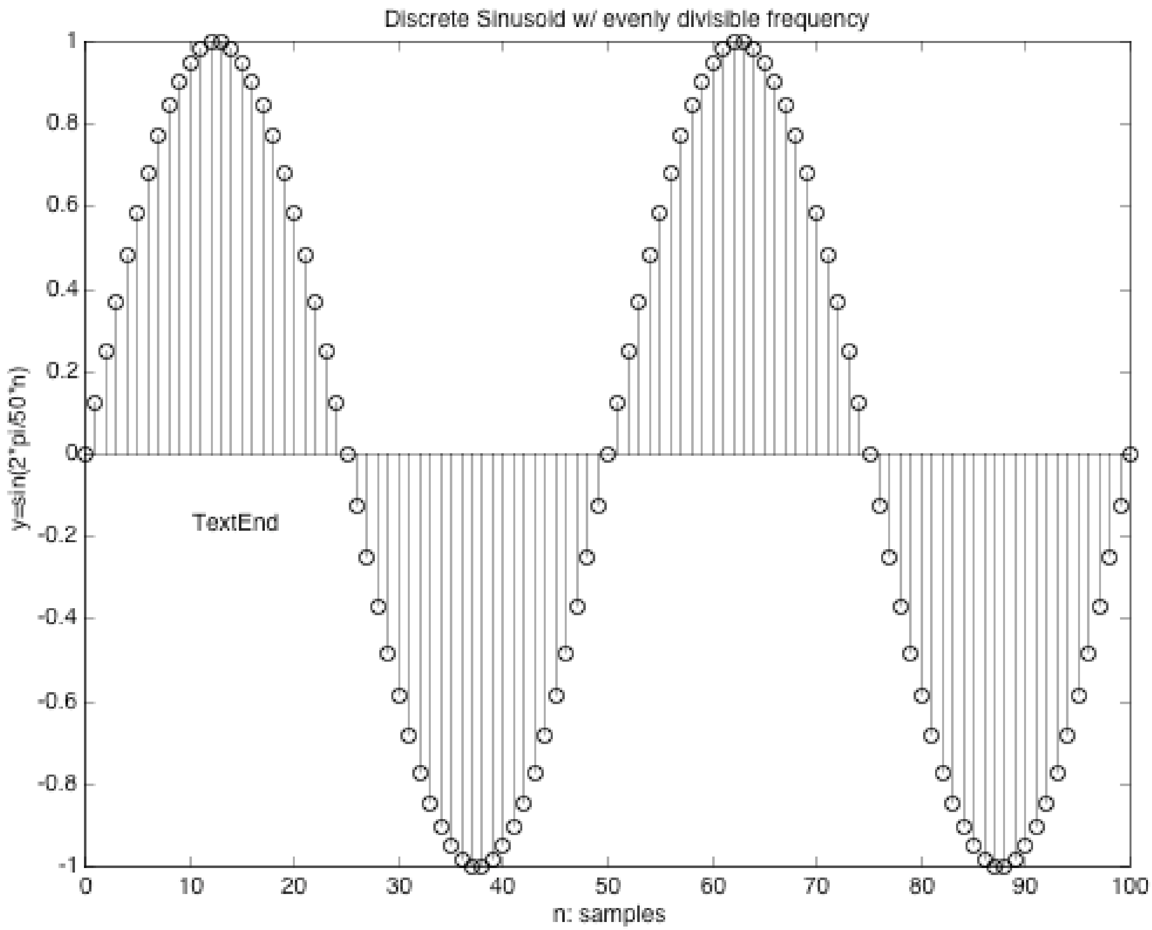
Smallest integer N such that $y[n]=y[n+N]$

Find an integer k so $N=k/f$ is also an integer

$$N \neq \frac{1}{f}$$

$f = k/N$ (f: rational number -> k/N is ratio of integers)

Period of a Discrete Sinusoid:ex1



$$y[n] = \sin(2\pi \cdot \tfrac{1}{50} \cdot n)$$

T=50 samples (integer)

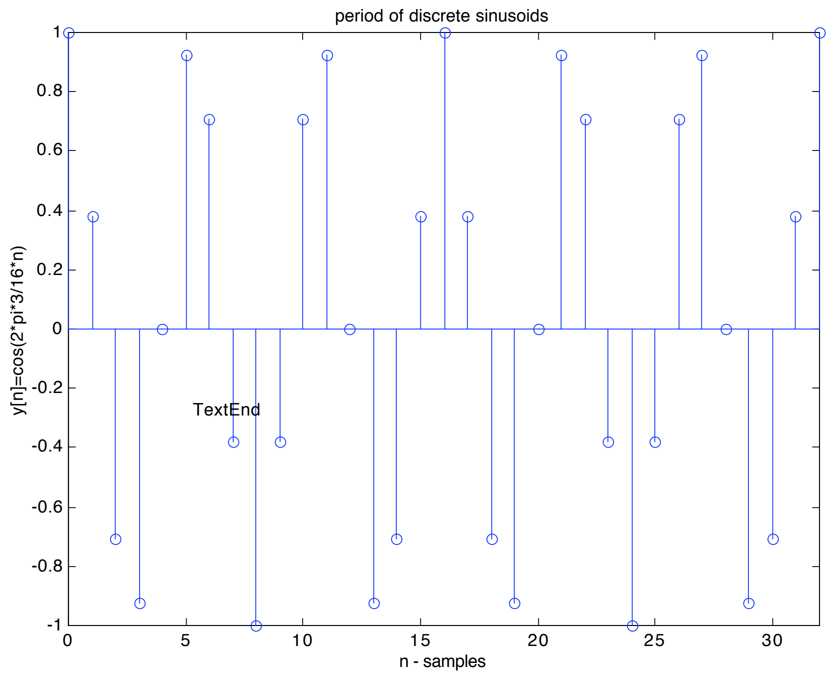
$$y[n] = y[n+50]$$

$$\sin(0) = \sin(2\pi)$$

Period of discrete sinusoids: ex2

ex: $y[n]=\cos(2\pi(3/16)n)$

What is the period N?



$y[n]=A\cos(2\pi fn+\phi)$

frequency: f=3/16 cycles/sample

N:period
(samples/repeating cycle [integer])

Smallest integer N such that $y[n]=y[n+N]$

Find an integer k so N=k/f is also an integer

f=3/16
let k=3
N=(3)*16/3=16

f = k/N (rational number -> k/N is ratio of integers)

Period of discrete sinusoids: ex3.

$$y[n] = \sin(2\pi \cdot \tfrac{3}{50} \cdot n)$$

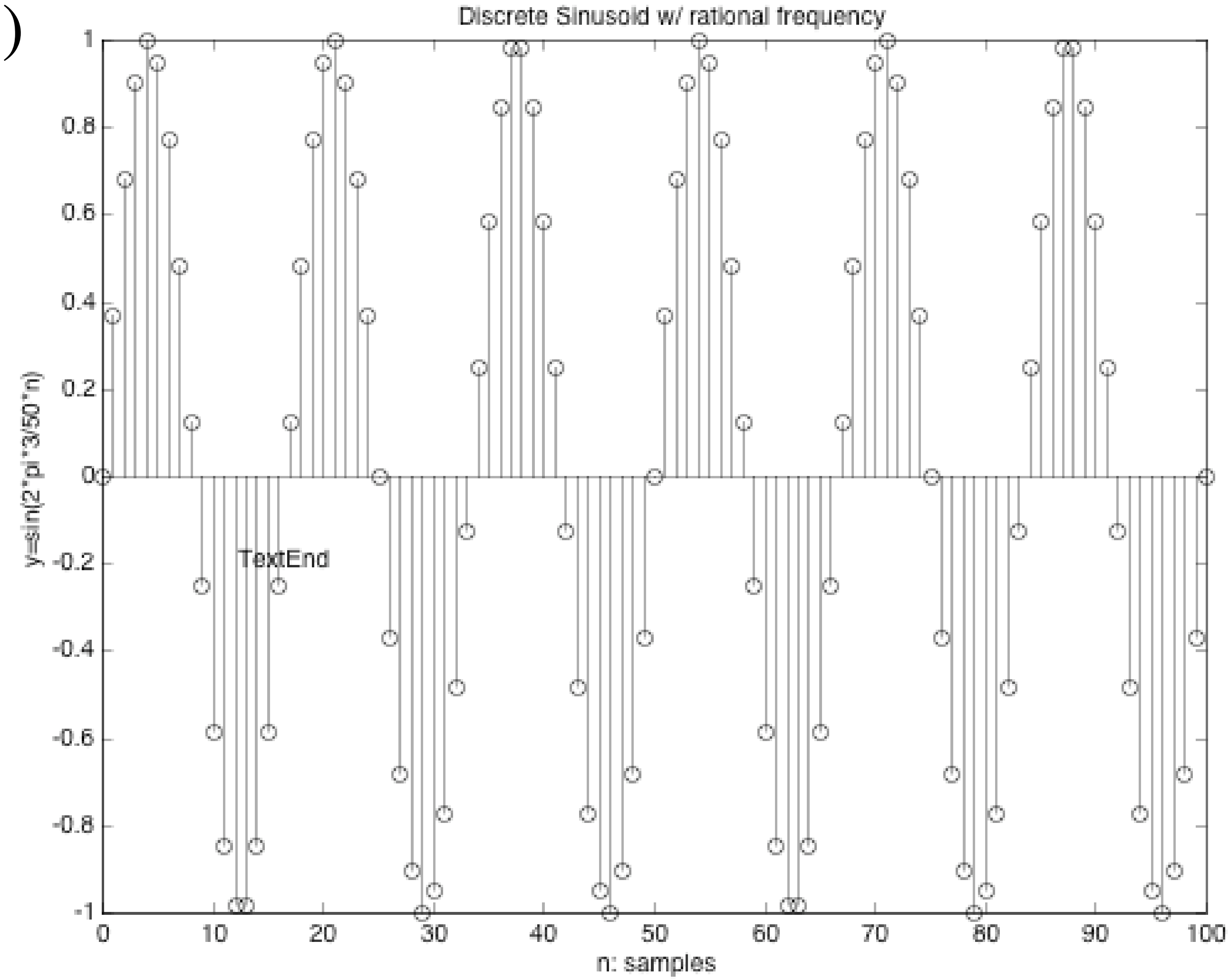
$$f = \tfrac{3}{50} \tfrac{cycles}{sample}$$

$$y[n] = y[n+N]$$

N=?? samples
N: integer

$$N \neq \tfrac{1}{f}$$

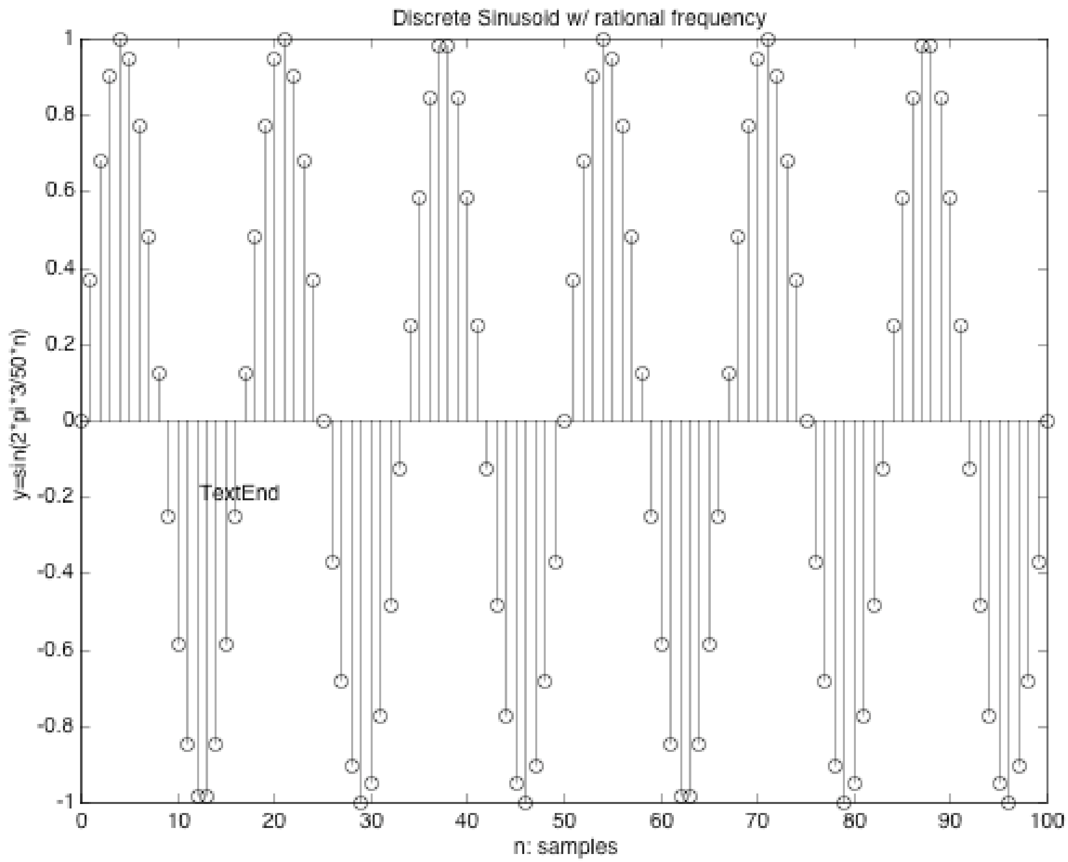
50/3 ≠ integer



Period of discrete sinusoids: ex3.

discrete function

$$y[n] = \sin(2\pi \cdot \tfrac{3}{50} \cdot n)$$



frequency: $f = \tfrac{3}{50} \frac{cycles}{\sec}$

period: N=?? samples

$$y[n] = y[n+N]$$

$f \cdot N = k$ N, k:integers

$$\tfrac{3}{50} \cdot N = k$$

$$\frac{N}{k} = \frac{50}{3} \frac{samples}{cycle}$$

ratio of integers
rational number

periodic
N=50 samples, k=3 cycles

Aperiodic discrete sinusoids

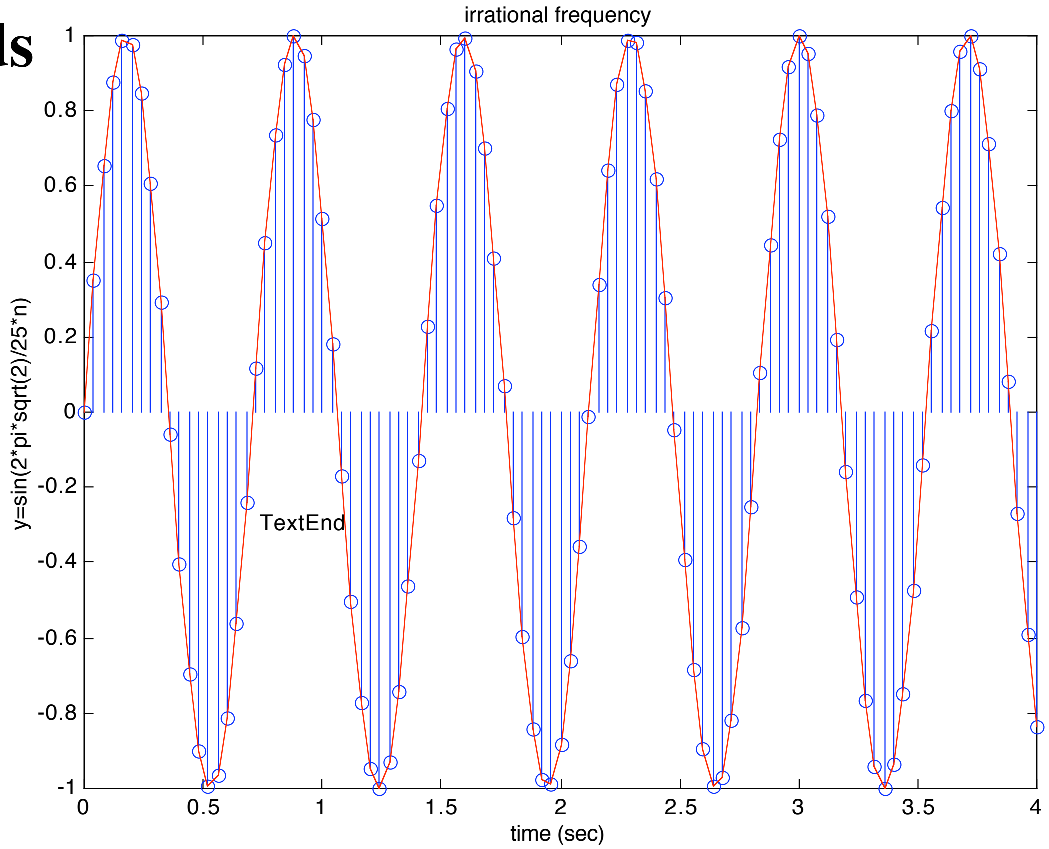
continuous function

$$y(t) = \sin(2\pi \cdot \sqrt{2} \cdot t)$$

$T = \frac{1}{\sqrt{2}}$ sec periodic

sample

$t = nT_s$ $T_s = \frac{1}{25}$ sec



discrete function

$$y[n] = \sin(2\pi \cdot \tfrac{\sqrt{2}}{25} \cdot n)$$

period?

$$y[n] = y[n+N]$$

N=?? samples (integer)

$$f = \tfrac{\sqrt{2}}{25}$$

$f \cdot N = k$ N,k: integers

$$\frac{N}{k} = \frac{25\sqrt{2}}{2}$$

not a ratio of integers
irrational number

sampled discrete sinusoid aperiodic

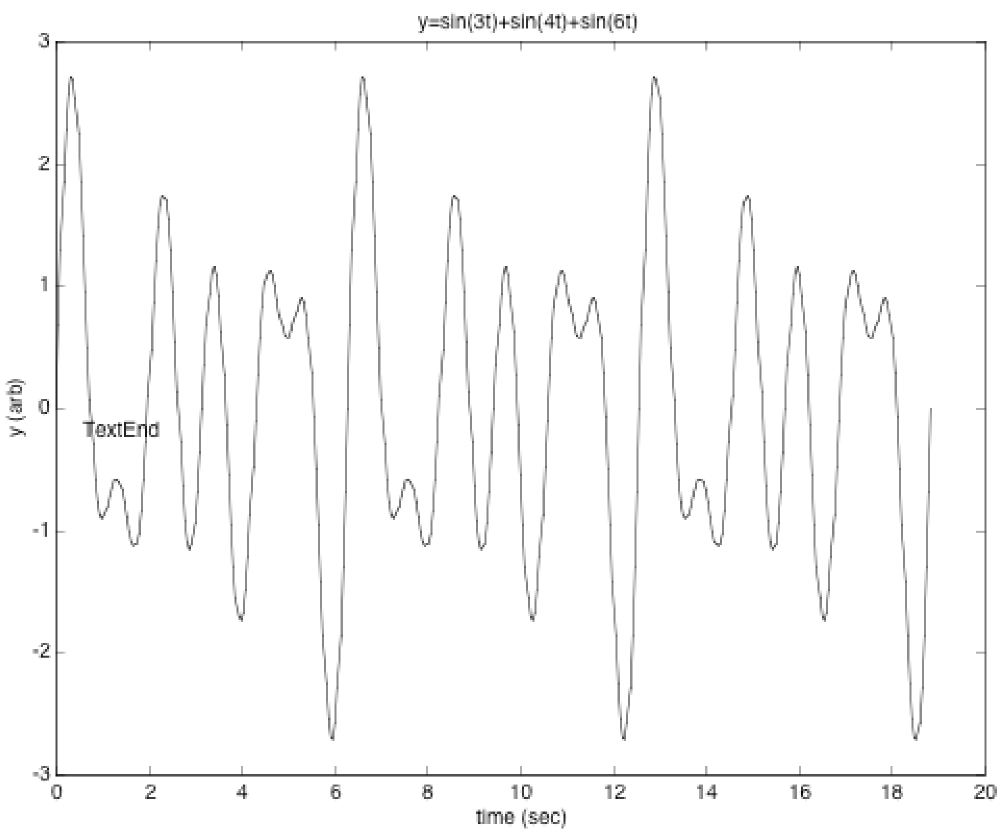
Periodicity

arbitrary continuous signal

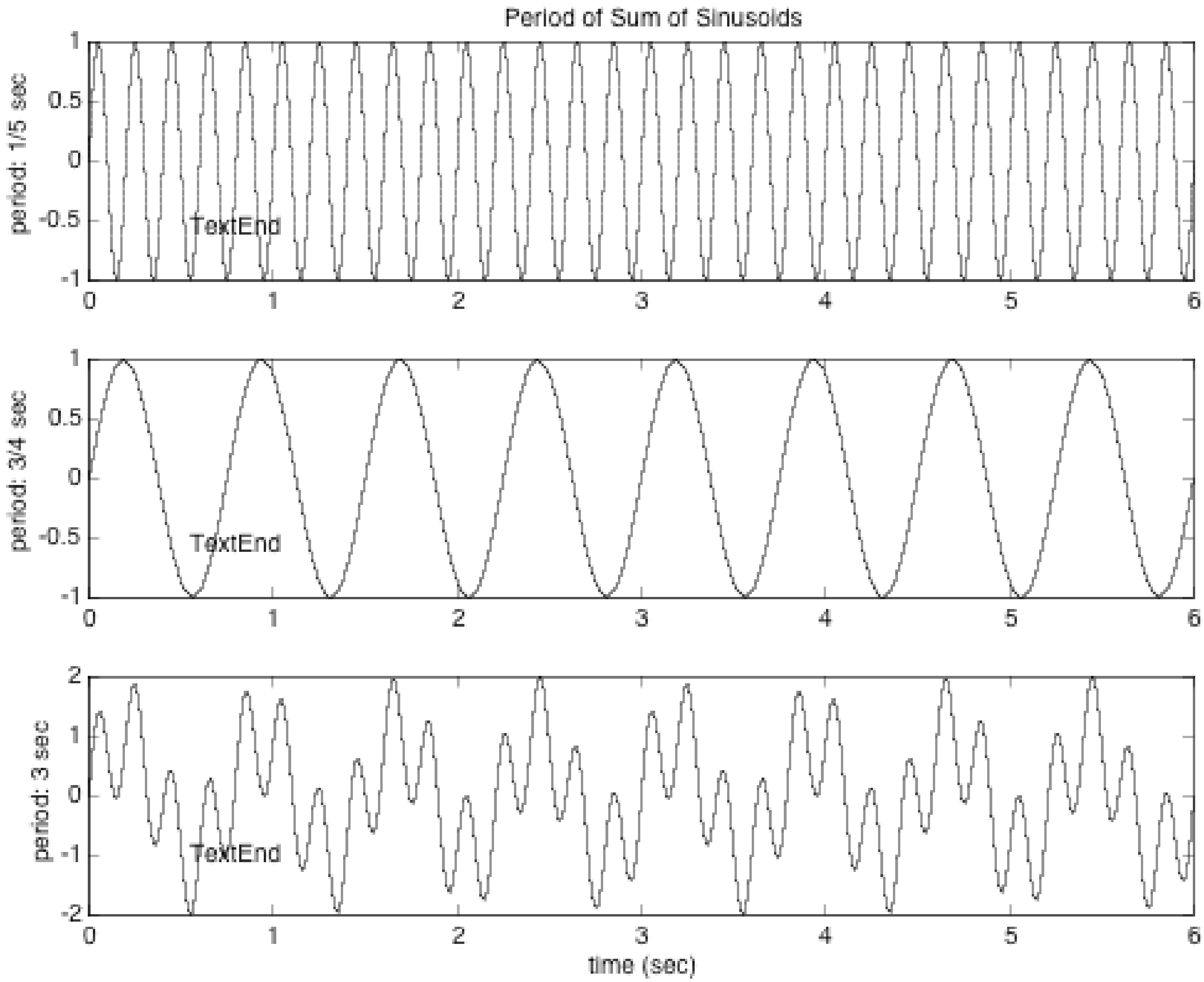
T:period (sec/cycle)

y(t+T)=y(t)

After what interval does the signal repeat itself?

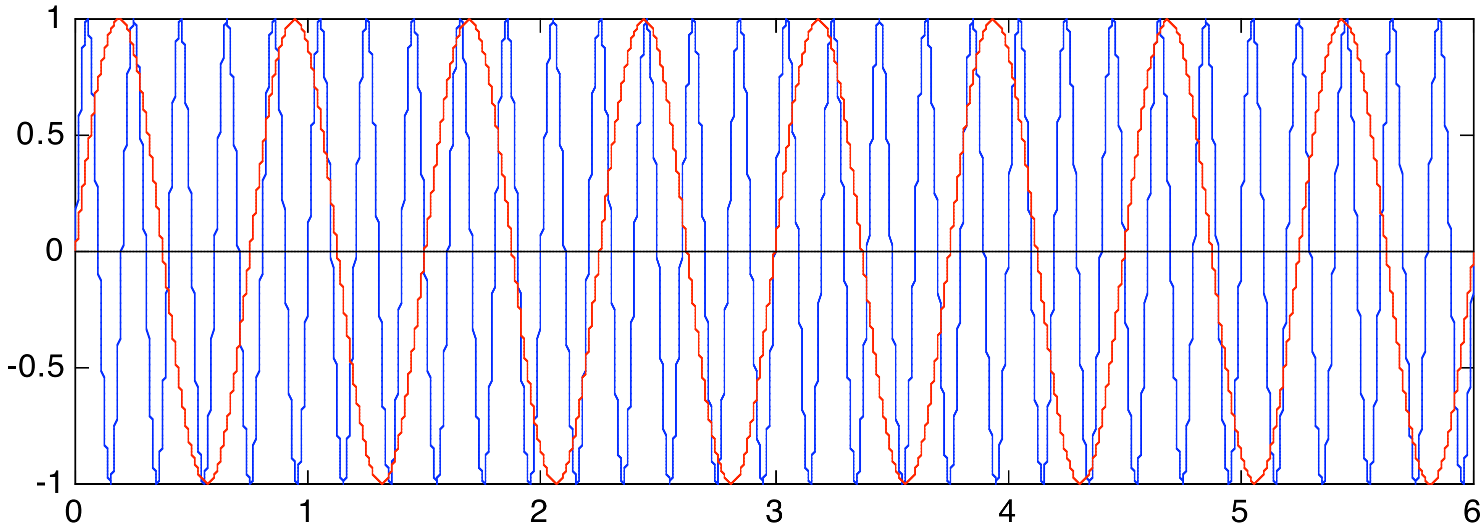


Period of Sum of Sinusoids

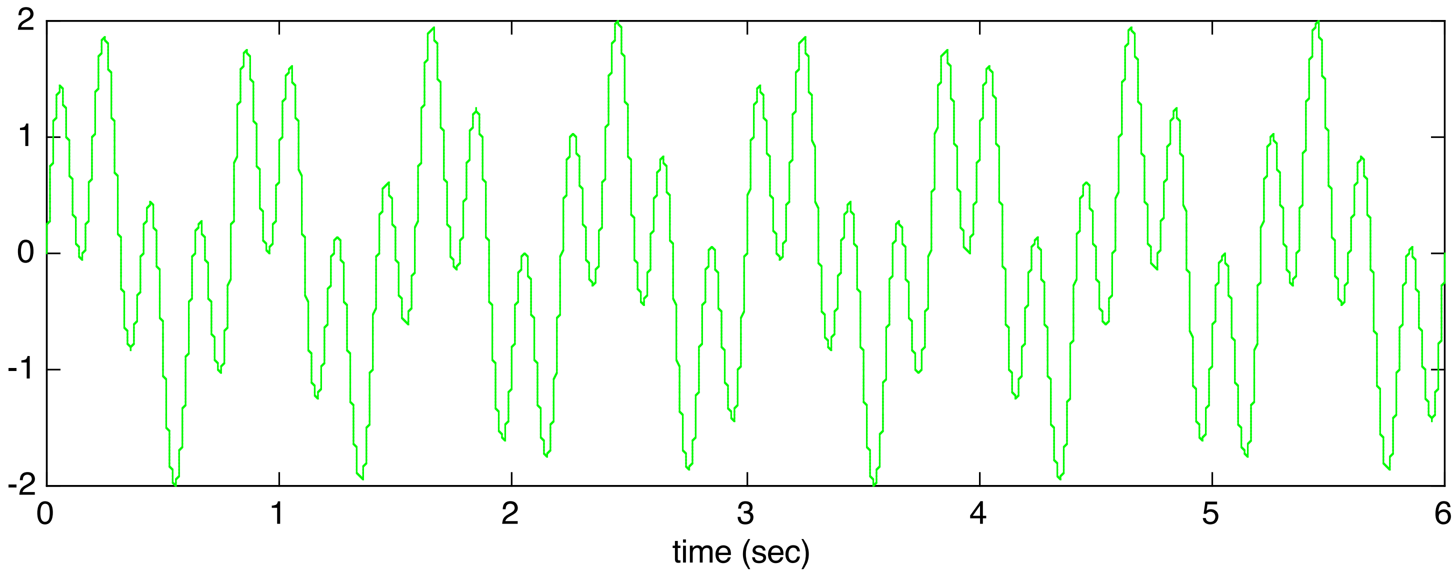


$$y(t) = y(t + T)$$

Ex: Period of sum of sinusoids



T1=0.2 seconds, T2=0.75 seconds



Tsum=? seconds

Least common multiple

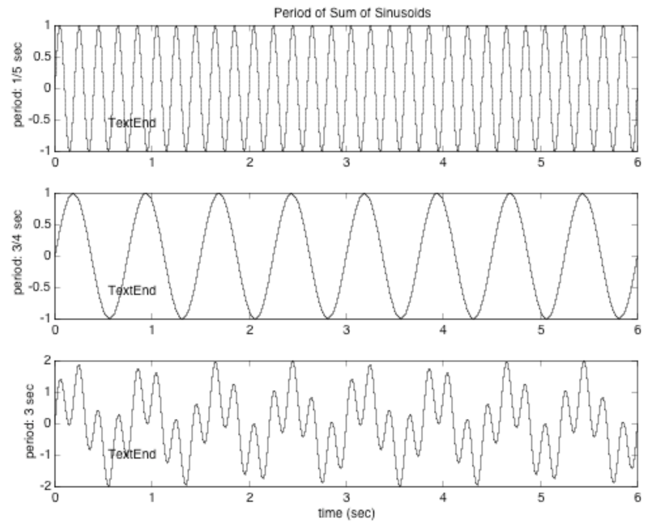
seconds to complete cycles

T1=0.2s=1/5 seconds

1/5s, 2/5s, 3/5s …

4/20s, 8/20s, 12/20s, 16/20s, 20/20s, 24/20s, 28/20s, 32/20s, 36/20s, 40/20s, 44/20s, 48/20s, 52/20s, 56/20s, 60/20s

15 cycles



seconds to complete cycles

T2=0.75s=3/4 seconds

3/4s, 6/4s, …

15/20s. 30/20s, 45/20s, 60/20s

4 cycles

1/5*k=3/4*l

k/l=15/4

rational number

Tsum=15*T1=15/5=3 seconds

Tsum=4*T2=3/4*4=3 seconds

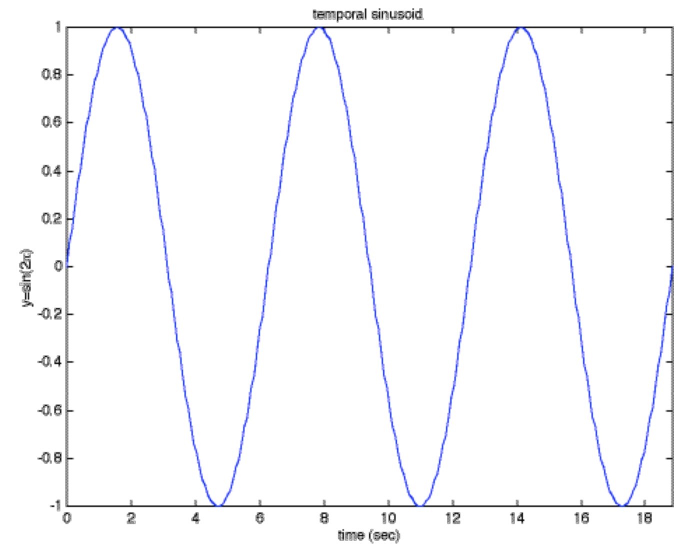
Tsum=3 seconds

Instantaneous frequency

$y(\theta)=\sin(\theta)$

$\theta=\theta(t)$

time varying argument



instantaneous frequency

$\omega=d\theta/dt$

sinusoid constant frequency

$y(t)=A\sin(\omega t+\phi)$

$\theta=\omega t+\phi$

$d\theta/dt=\omega$

chirp linearly swept frequency

$\omega=d\theta/dt$

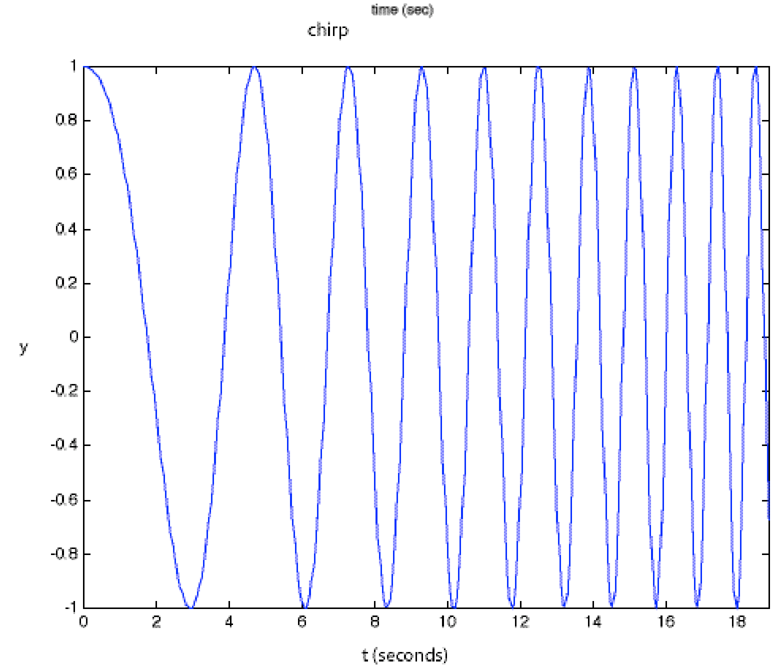
$\omega=((\omega_1-\omega_0)/T)t+\omega_0$

t	ω
0	ω_0
T	ω_1

integrate

$\theta=(\omega_1-\omega_0)/2T\, t^2+\omega_0 t + C$

$y_{chirp}(t)=A\sin((\omega_1-\omega_0)/(2T)\, t^2+\omega_0 t+\phi)$



Representations of a sinusoid

$y(t)=A\cos(\omega t+\phi)$ trig function

$$y(t) = Ae^{j\phi}\frac{\left[e^{j\omega t} + e^{-j\omega t}\right]}{2}$$ complex conjugates

$$e^{j(\theta)} = e^{j(\omega t+\phi)}$$
$$= e^{j\phi}e^{j\omega t}$$

$y(t)=\text{Re}\{Ae^{j\phi}e^{j(\omega t)}\}$ real part of complex exponential

Add spectrum

$X= Ae^{j\phi}$ complex amplitude (constant)

$y(t)=\text{Re}\{Xe^{j(\omega t)}\}$ rotating phasor

Euler's relations

$e^{j\theta}=\cos(\theta)+j\sin(\theta)$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Conversions

cartesian → polar

s=a+jb

$$s = \sqrt{a^2 + b^2}\, e^{j \cdot a\tan\left(b/a\right)}$$

polar → cartesian

s=re$^{j\theta}$

$$s = r\cos\theta + jr\sin\theta$$

Complex Arithmetic		
Addition	cartesian	$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$
Subtraction	cartesian	$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$
Multiplication	polar	$r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
Division	polar	$\dfrac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \dfrac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
Powers	polar	$\left(re^{j\theta}\right)^n = r^n e^{jn\theta}$
Roots	polar	$z^n = s = re^{j\theta}$ $z = s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)}$ $k = 1, 2 \ldots n-1$

Complex Exponentials

Why use complex exponentials?

Trigonometric manipulations -> algebraic operations on exponents

Trigonometric identities

$$\cos(x)\cos(y) = \tfrac{1}{2}\left[\cos(x-y) - \left[\cos(x+y)\right]\right]$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Properties of exponentials

$re^x e^y = re^{x+y}$

$(re^x)^n = r^n e^{nx}$

$$\sqrt[n]{x} = x^{1/n}$$

$$\frac{1}{x} = x^{-1}$$

Vector representation (graphical)

Complex Exponentials

Why use complex exponentials?

Trigonometric manipulations -> algebraic operations on exponents

Adding sinusoids of same frequency but multiple amplitudes and phases

$$A\cos(\omega t+\phi_1)+B\cos(\omega t+\phi_2)=C\cos(\omega t+\phi_3)$$

$$A[\cos(\omega t)\cos(\phi_1)-\sin(\omega t)\sin(\phi_1)]+B[\cos(\omega t)\cos(\phi_2)-\sin(\omega t)\sin(\phi_2)]$$

*sum formula for sin & cos trig id

$$-[A\sin(\phi_1)+B\sin(\phi_2)]\sin(\omega t)+[A\cos(\phi_1)+B\cos(\phi_2)]\cos(\omega t)$$

$$\sqrt{A^2+2\cdot A\cdot B\cdot\cos(\phi 1-\phi 2)+B^2}\cdot\sin\left[\omega\cdot t+\operatorname{atan}\left[\frac{(A\cdot\cos(\phi 1)+B\cdot\cos(\phi 2))}{(-A\cdot\sin(\phi 1)-B\cdot\sin(\phi 2))}\right]\right]$$

or

$$A\cdot\cos(\omega+\phi 1)+B\cdot\cos(\omega+\phi 2)$$

Re( )=cos

$$\mathrm{Re}\cdot\left(A\cdot e^{j\phi 1}\cdot e^{j\cdot\omega\cdot t}+B\cdot e^{j\cdot\phi 2}\cdot e^{j\cdot\omega\cdot t}\right)$$

Adding complex amplitudes

$$\mathrm{Re}\cdot\left[\left(A\cdot e^{j\phi 1}+B\cdot e^{j\cdot\phi 2}\right)\cdot e^{j\cdot\omega\cdot t}\right]$$

cos + jsin

$$\mathrm{Re}\cdot\left[\left[A\cdot(\cos(\phi 1)+j\cdot\sin(\phi 1))+B\cdot(\cos(\phi 2)+j\cdot\sin(\phi 2))\right]\cdot e^{j\cdot\omega\cdot t}\right]$$

$$\mathrm{Re}\cdot\left[\left[A\cdot\cos(\phi 1)+B\cdot\cos(\phi 2)+j\cdot(A\cdot\sin(\phi 1)+B\cdot\sin(\phi 2))\right]\cdot(\cos(\omega\cdot t)+j\cdot\sin(\omega\cdot t))\right]$$

$$\mathrm{Re}\cdot\begin{bmatrix}\left[-(A\cdot\sin(\phi 1)+B\cdot\sin(\phi 2))\cdot\sin(\omega\cdot t)+(A\cdot\cos(\phi 1)+B\cdot\cos(\phi 2))\cdot\cos(\omega\cdot t)+\blacksquare\right]\\ \left[(A\cdot\cos(\phi 1)+B\cdot\cos(\phi 2))\cdot\sin(\omega\cdot t)+(A\cdot\sin(\phi 1)+B\cdot\sin(\phi 2))\cdot\cos(\omega\cdot t)\right]\cdot j\end{bmatrix}$$

$$\left[-(A\cdot\sin(\phi 1)+B\cdot\sin(\phi 2))\cdot\sin(\omega\cdot t)+(A\cdot\cos(\phi 1)+B\cdot\cos(\phi 2))\cdot\cos(\omega\cdot t)\right]$$

Complex Exponentials

Amplitude modulation (multiply two sinusoids of different frequencies)

$$A\cos(\omega_1 t)\ B\cos(\omega_2 t+\phi) = C(\cos(\omega_3 t+\phi_2)+\cos(\omega_4 t+\phi_2))$$

$$A\cdot\cos(\omega 1)\cdot(B\cdot\cos(\omega 2+\phi 2))$$

$$A\cdot B\cdot\cos(\omega 1)\cdot((\cos(\omega 2)\cdot\cos(\phi 2)-\sin(\omega 2)\cdot\sin(\phi 2)))$$

*sum formula for sin & cos trig id

$$A\cdot B\cdot\cos(\omega 1)\cdot\cos(\phi 2)-A\cdot B\cdot\cos(\omega 1)\cdot\sin(\omega 2)\cdot\sin(\phi 2)$$

*product formula for sin & cos trig id

$$A\cdot B\cdot\frac{\cos(\omega 2+\omega 1)+\cos(\omega 2-\omega 1)}{2}\cdot\cos(\phi 2)-A\cdot B\cdot\frac{\sin(\omega 2+\omega 1)+\sin(\omega 2-\omega 1)}{2}\cdot\sin(\phi 2)$$

$$A\cdot B\cdot\frac{\cos(\omega s)+\cos(\omega d)}{2}\cdot\cos(\phi 2)-A\cdot B\cdot\frac{\sin(\omega s)+\sin(\omega d)}{2}\cdot\sin(\phi 2)$$

*sum formula for sin & cos trig id

$$\frac{1}{2}\cdot A\cdot B\cdot(\cos(\phi 2)\cdot\cos(\omega s)+\cos(\phi 2)\cdot\cos(\omega d)-\sin(\phi 2)\cdot\sin(\omega s)-\sin(\phi 2)\cdot\sin(\omega d))$$

$$\frac{1}{2}\cdot A\cdot B\cdot(\cos(\omega s+\phi 2)+\sin(\omega d+\phi 2))$$

or

$$A\cdot\cos(\omega 1)\cdot(B\cdot\cos(\omega 2+\phi 2))$$

cos = complex conj

$$A\cdot\left[\frac{e^{j\cdot\omega 1}+e^{-(j\cdot\omega 1)}}{2}\right]\cdot\left[B\cdot\left[\frac{e^{j\cdot(\omega 2+\phi)}+e^{-j\cdot(\omega 2+\phi)}}{2}\right]\right]$$

mult. exponentials=add exponents

$$\frac{1}{4}\cdot A\cdot B\cdot[\exp[j\cdot(\phi+\omega 2+\omega 1)]+\exp[-j\cdot(\phi+\omega 2+\omega 1)]+\exp[-j\cdot(\phi+\omega 2-\omega 1)]+\exp[j\cdot(\phi+\omega 2-\omega 1)]]$$

cos = complex conj

$$\frac{1}{4}\cdot A\cdot B\cdot(2\cdot\cos(\phi+\omega 2+\omega 1)+2\cdot\cos(\phi+\omega 2-\omega 1))$$

$$\frac{1}{2}\cdot A\cdot B\cdot(\cos(\phi+\omega 2+\omega 1)+\cos(\phi+\omega 2-\omega 1))$$

Representations of Sinusoids

$$A\cos(2\pi k f_0 t + \phi_k)$$

$$\mathrm{Re}\left\{Ae^{j2\pi ft+\phi}\right\}$$
$$= \mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\}$$
$$= \mathrm{Re}\left\{Xe^{j2\pi ft}\right\}$$

$$Ae^{j\phi}\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$
$$= X\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$

Sum multiple cosines same frequency

$$\sum_{k=1}^{n} A_k\cos(2\pi ft+\phi_k) = \sum_{k=1}^{n}\mathrm{Re}\left\{A_k e^{2\pi ft+\phi_k}\right\} = \sum_{k=1}^{n}\mathrm{Re}\left\{A_k e^{\phi_k}e^{2\pi ft}\right\}$$
$$= \left(\sum_{k=1}^{n}\mathrm{Re}\left\{A_k e^{\phi_k}\right\}\right)e^{2\pi ft}$$

sum complex amplitudes

Ex. $3\cos\left(2\pi 40t+\frac{\pi}{2}\right)-1\cos\left(2\pi 40t-\frac{\pi}{6}\right)+2\cos\left(2\pi 40t+\frac{\pi}{3}\right)$

$$\mathrm{Re}\left\{3e^{j\frac{\pi}{2}}e^{j2\pi 40t}-1e^{-j\frac{\pi}{6}}e^{j2\pi 40t}+2e^{\frac{\pi}{3}}e^{j2\pi 40t}\right\}$$

$$\mathrm{Re}\left\{\left(3e^{j\frac{\pi}{2}}-1e^{-j\frac{\pi}{6}}+2e^{\frac{\pi}{3}}\right)e^{j2\pi 40t}\right\}$$

$$\mathrm{Re}\left\{5.234e^{j1.545}e^{2\pi 40t}\right\}$$

$$5.234\cos(2\pi 40t+1.545)$$

Multiply cosines of different frequencies

$$A_1 \cos(\omega_1 t) \cdot A_2 \cos(\omega_2 t + \phi)$$

$$A_1 \left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2} \right) A_2 \left(\frac{e^{j(\omega_2 t + \phi)} + e^{-j(\omega_2 t + \phi)}}{2} \right)$$

$$\frac{A_1 A_2}{4} \left(e^{j\omega_1 t} e^{j(\omega_2 t + \phi)} + e^{j\omega_1 t} e^{-j(\omega_2 t + \phi)} + e^{-j\omega_1 t} e^{j(\omega_2 t + \phi)} + e^{-j\omega_1 t} e^{-j(\omega_2 t + \phi)} \right)$$

$$\frac{A_1 A_2}{4} \left(e^{j(\omega_1 t + \omega_2 t + \phi)} + e^{-j(\omega_2 t - \omega_1 t + \phi)} + e^{j(\omega_2 t - \omega_1 t + \phi)} + e^{-j(\omega_1 t + \omega_2 t + \phi)} \right)$$

$$\frac{A_1 A_2}{2} \left(\cos((\omega_1 + \omega_2)t + \phi) + \cos((\omega_2 - \omega_1)t + \phi) \right)$$

Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

synthesize a periodic signal x(t) from a sum of a series of sinusoids - the Fourier series.

$f_k = kf_0$, k integer f_0 fundamental frequency
f_k harmonic frequencies

Note: The sum of harmonic sinusoids is periodic with a period equal to the fundamental period.

ex.

$$X_k = \begin{cases} \dfrac{-8}{\pi^2 k^2} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$f_0 = 25Hz$$

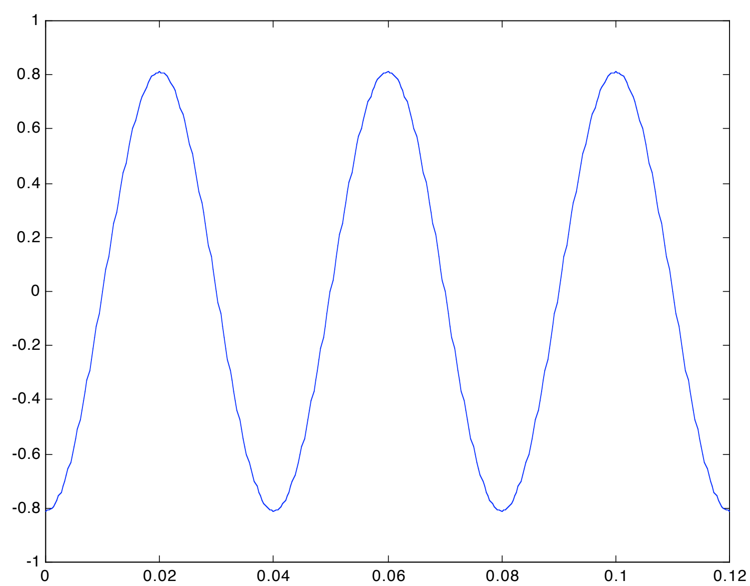
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{-8}{\pi^2 k^2} & k \quad odd \\ 0 & k \quad even \end{cases} \qquad X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=1 $\quad X_1 = -8/(1\pi^2) = -0.8105 \quad f_1 = k f_0 = 1 \cdot 25 = 25$

$$x(t) = 0.8105 \cos\left(2\pi 25 t + \pi\right)$$



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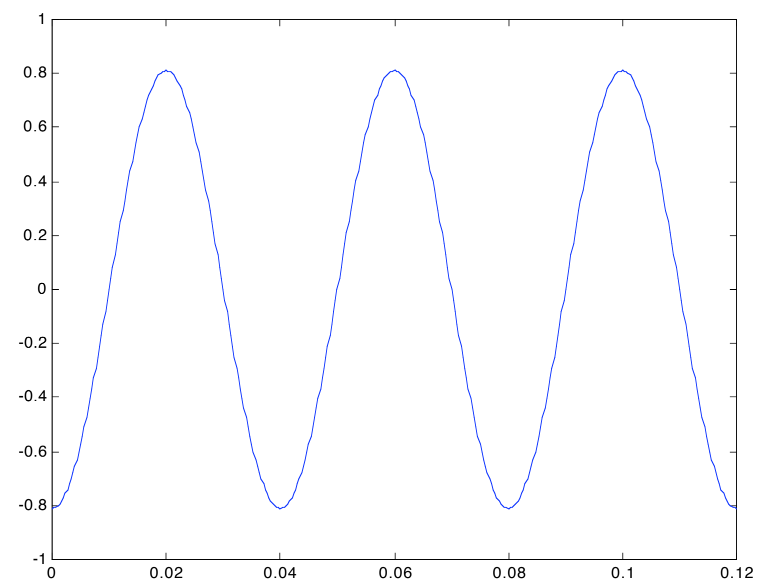


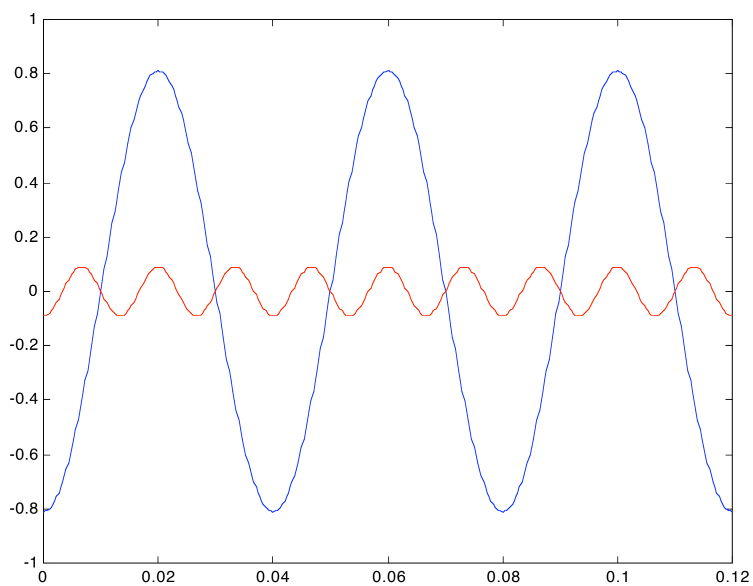
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=3 $\quad X_3 = -8/(3^2\pi^2) = -0.8105 \quad f_3 = kf_0 = 3\cdot 25 = 75$

$$x(t) = 0.8105\cos\left(2\pi 25t + \pi\right) + 0.0901\cos\left(2\pi 75t + \pi\right)$$



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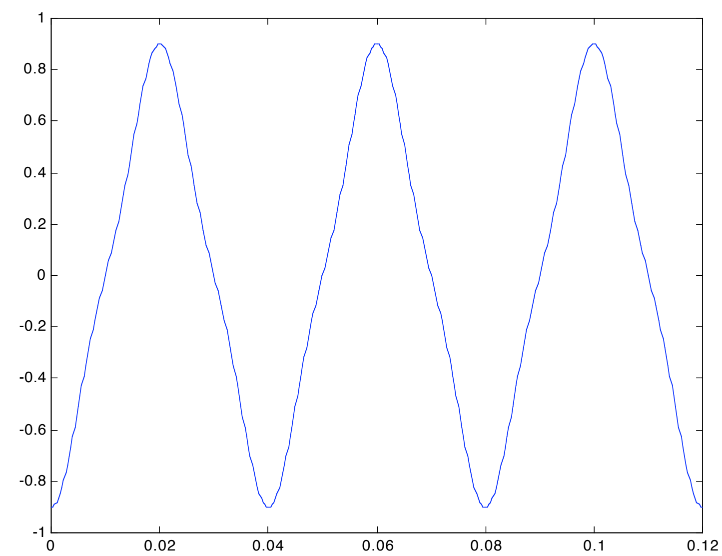


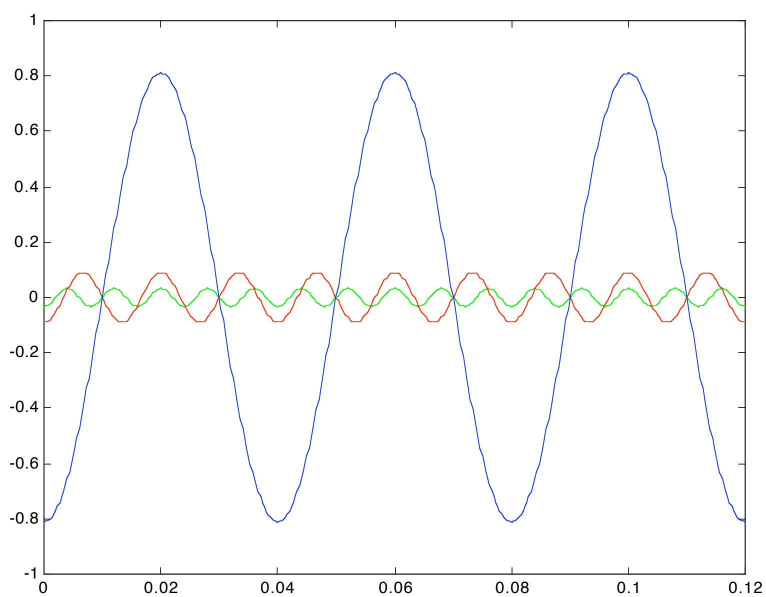
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=5 $\quad X_5 = -8/(5^2\pi^2) = -0.0324 \qquad f_5 = k f_0 = 5\cdot 25 = 125$

$$x(t) = 0.8105\cos\left(2\pi 25 t + \pi\right) + 0.0901\cos\left(2\pi 75 t + \pi\right) + 0.0324\cos\left(2\pi 125 t + \pi\right)$$



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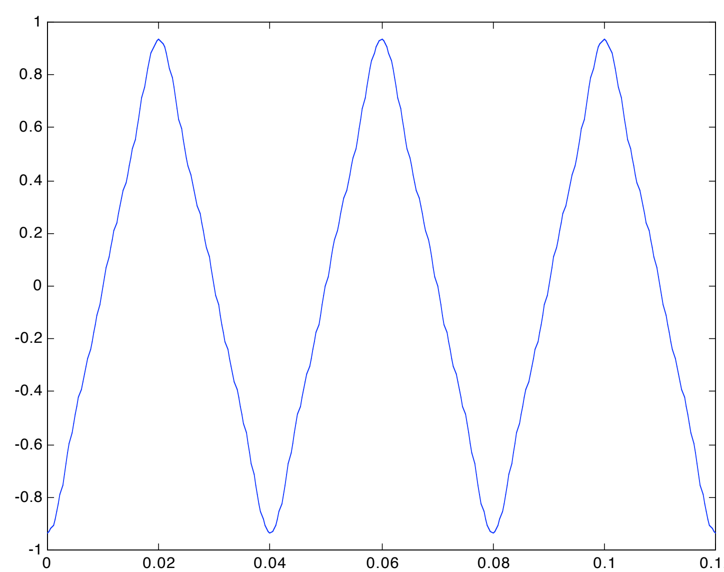
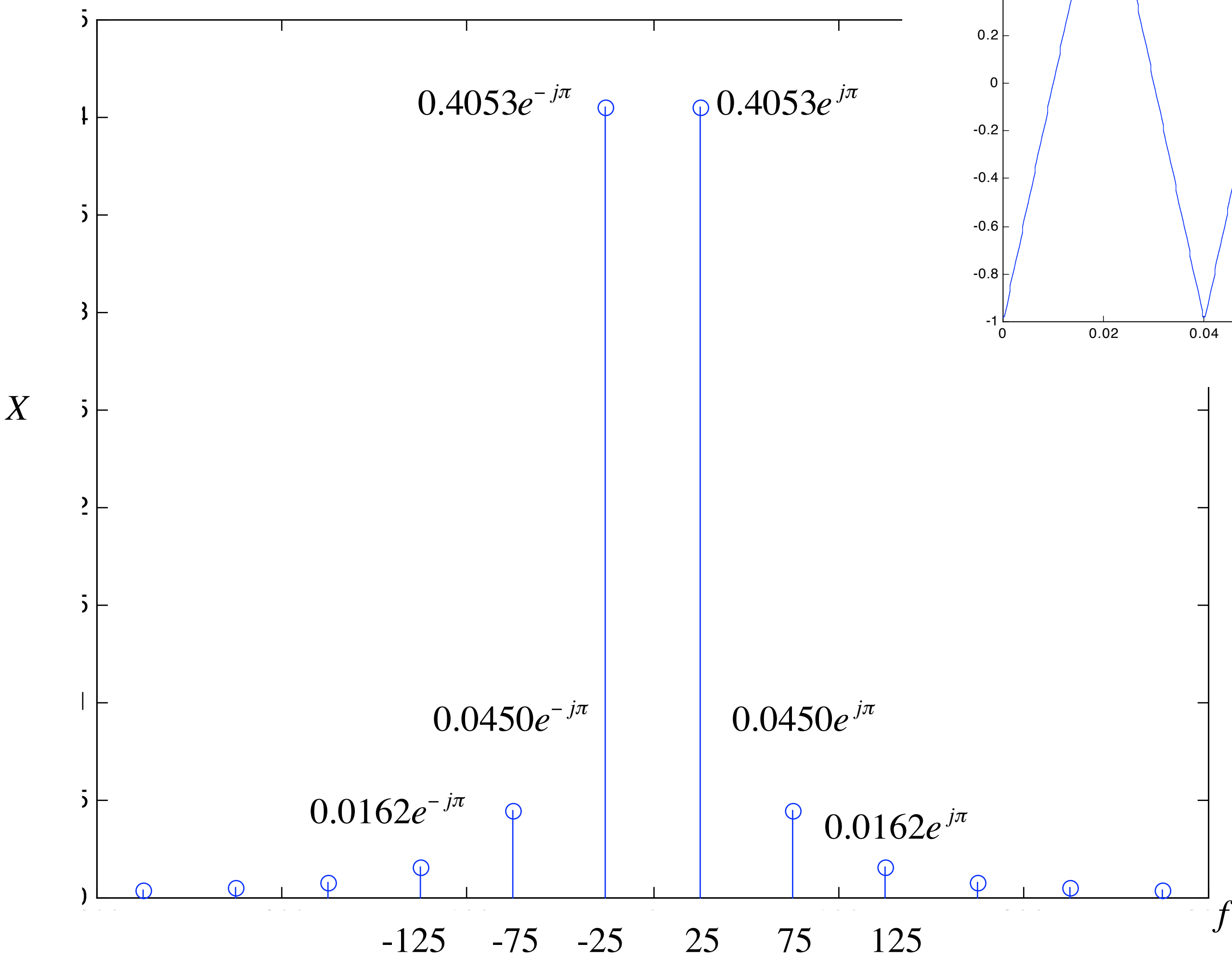
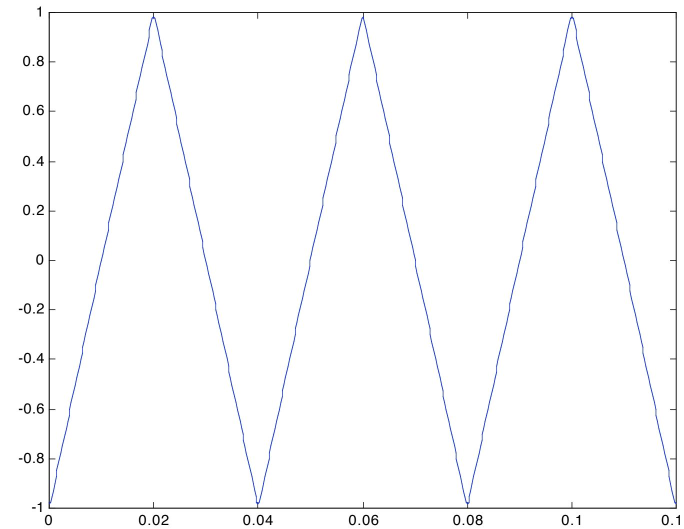


spectrum





$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi) + \ldots$$

$$x(t) = \frac{0.8105}{2} e^{\pi}\left(e^{j2\pi 25t} + e^{-j2\pi 25t}\right) + \frac{0.0901}{2} e^{\pi}\left(e^{j2\pi 75t} + e^{-j2\pi 75t}\right) + \frac{0.0324}{2} e^{\pi}\left(e^{j2\pi 125t} + e^{-j2\pi 125t}\right) + \ldots$$

Fourier Series

For a given signal, how do we find $X_k = A_k e^{j\phi_k}$ for each k ?

Fourier Analysis

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

where

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

f_0:fundamental frequency

$T_0 = 1/f_0$

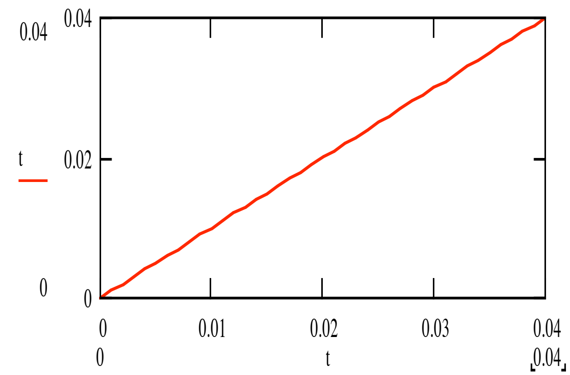
$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \text{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$



$$= \frac{1}{T_0}\int_0^{T_0} t dt = \frac{1}{T_0}\frac{t^2}{2}\bigg|_0^{T_0} = \frac{1}{T_0}\frac{T_0^{\,2}}{2} = \frac{T_0}{2}$$

Mathematica:

athena%add math

athena%math

In[1]:=1/T*Integrate[t,{t,0,T}]

Out[1]:=T/2

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{2}{T_0}\int_0^{T_0} t e^{-j2\pi kt/T_0} dt$$

$$X_k = \frac{T_0}{2}\frac{\left(j2\pi k + 1\right)}{\pi^2 k^2} e^{-2j\pi k} - \frac{T_0}{2\pi^2 k^2}$$

$$e^{-2j\pi k} = \left(e^{-j2\pi}\right)^k$$

$$= 1^k$$

$$= 1$$

$$X_k = \frac{T_0}{2}\frac{\left(j2\pi k + 1\right)}{\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j\frac{T_0}{2}\frac{2k\pi}{\pi^2 k^2} + \frac{T_0}{2\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j\frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

Fourier Series

Mathematica:

```
In[2]:= 2/T*Integrate[t*Exp[-I*2*Pi*k*t/T],{t,0,T}]
```

$$X_k = \frac{2}{T_0}\int_0^{T_0} t e^{-j2\pi kt/T_0} dt$$

```
                 (2 I) k Pi
       -((-1 + E            - (2 I) k Pi) T)
Out[2]= ------------------------------------
                (2 I) k Pi  2   2
            2 E            k  Pi
```

```
In[3]:= Simplify[%,Element[k,Integers]]
```

$$e^{-2j\pi k} = 1$$

$$e^{-j\pi k} = -1^k$$

```
        I T
Out[3]= ----
        k Pi
```

$$X_k = j\frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_0 = \frac{T_0}{2}$$

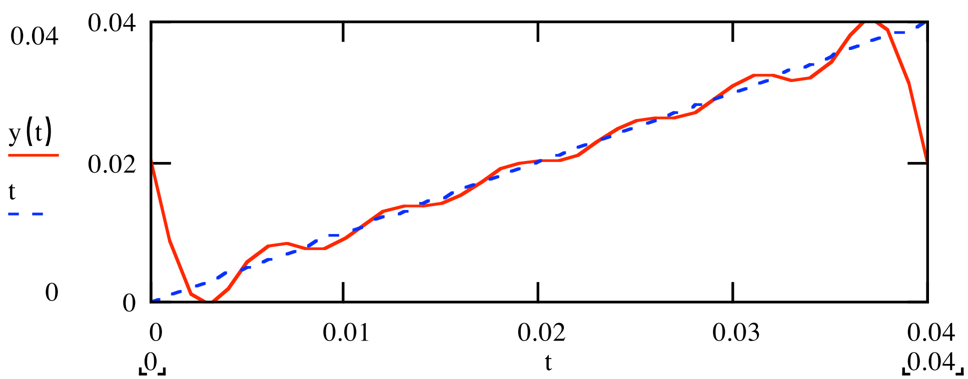
$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

f_0:fundamental frequency

$T_0 = 1/f_0$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos\left(2\pi f_0 t + \tfrac{\pi}{2}\right) + \frac{T_0}{2\pi} \cos\left(2\pi 2 f_0 t + \tfrac{\pi}{2}\right) + \ldots$$



$f_0 = 25Hz$

$T_0 = 1/f_0 = 0.04$

7 terms

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

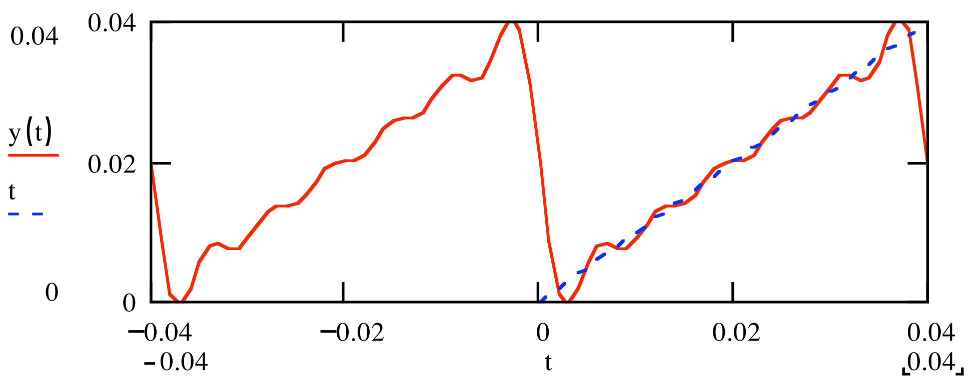
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j 2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos\left(2\pi f_0 t + \tfrac{\pi}{2}\right) + \frac{T_0}{2\pi} \cos\left(2\pi 2 f_0 t + \tfrac{\pi}{2}\right) + \ldots$$

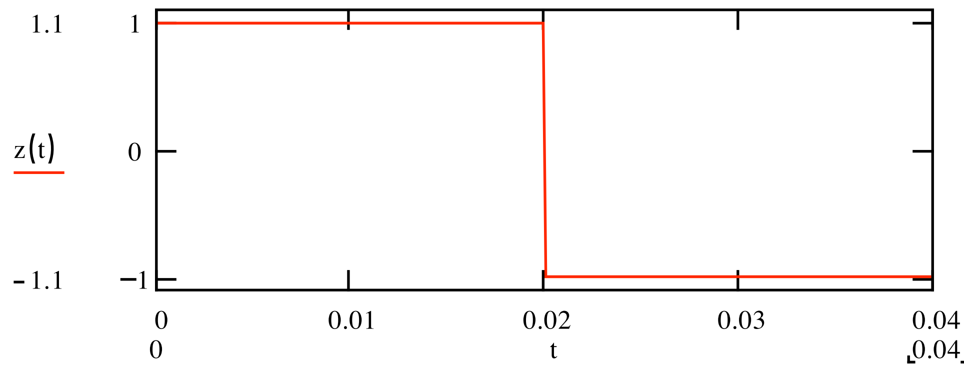


Defined between 0<t<0.04

Periodic with period 0.04

Fourier Series:Square Wave

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$



$$X_0 = \frac{1}{T_0}\int_0^{T_0/2} 1\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -1\,dt$$

In[1]:=1/T*Integrate[1,{t,0,T/2}]+ 1/T*Integrate[-1,{t,T/2,T}]
Out[1]:=0

$$X_0 = 0$$

Fourier Series:Square Wave

$$X_k = \frac{2}{T_0}\int_0^{T_0/2} 1e^{-j2\pi kt/T_0}dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} -1e^{-j2\pi kt/T_0}dt$$

```
In[2]:=2/T*Integrate[Exp[-I*2*Pi*k*t/T],{t,0,T/2}]+
        2/T*Integrate[- Exp[-I*2*Pi*k*t/T],{t,T/2,T}]
                   -I k Pi                 I k Pi
        -I (1 - E          )   I (-1 + E          )
Out[2]= ------------------ + ----------------
        k Pi                    (2 I) k Pi
                              E              k Pi
```

```
In[3]:= Simplify[%,Element[k,Integers]]

                       k 2
        -I (-1 + (-1) )
Out[7]= ----------------
               k Pi
```

$$X_k = \frac{-j\left(-1+(-1)^k\right)^2}{k\pi}$$

$$X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \frac{-j(-1-1)^2}{k\pi} = -\frac{j(-2)^2}{k\pi}$$

$$= -\frac{j4}{k\pi}$$

$$X_k = \frac{-j(-1+1)^2}{k\pi}$$

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

$$X_0 = 0$$

$$X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases} \longrightarrow X_k = \begin{cases} \dfrac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$