Yet more notes on partial fraction expansion
Revised Edition

The most general case of partial fraction expansion assumes you have a proper fraction with all positive powers of z (if it's not proper, carry out long division until it is; if it has negative powers multiply by z^n/z^n, where $-n$ is the highest negative power).

Let's suppose we've satisfied the above requirements, then we have an $X(z)$ that can be written as

$$X(z) = \frac{B(z)}{A(z)},$$

where the numerator is of order m and the denominator of order n, $m \leq n$. The first step is to factor the denominator:

$$X(z) = \frac{B(z)}{(z - p_1)(z - p_2) \ldots (z - p_n)},$$

where the p_i are poles of $X(z)$. Then we need to expand this into a sum of terms

$$X(z) = c_0 + \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \ldots + \frac{c_n z}{z - p_n},$$

the inverse z-transform of each of which we can look up in a table.

The individual c_i factors are solved for as follows:

$$c_0 = X(z) \mid_{z=0},$$

which is actually just the ratio of the constant terms in the numerator and denominator, while for $0 < i \leq n$

$$c_i = \left[\frac{z - p_i}{z} X(z) \right]_{z=p_i}.$$

Example: Find $x[n]$ when

$$X(z) = \frac{3 - \frac{5}{2} z^{-1}}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}.$$

First, we need to multiply through by z^2/z^2 to make all the powers positive, leaving us with

$$X(z) = \frac{3z^2 - \frac{5}{2} z}{z^2 - \frac{3}{2} z + \frac{1}{2}}.$$

Then we factor the denominator:

$$X(z) = \frac{3z^2 - \frac{5}{2} z}{(z - \frac{1}{2})(z - 1)}.$$

Now solve for the c_i:

$$c_0 = \frac{0}{\frac{1}{2}} = 0$$

$$c_1 = \left[\frac{z-\frac{1}{2}}{z}\frac{3z^2-\frac{5}{2}z}{(z-\frac{1}{2})(z-1)}\right]_{z=\frac{1}{2}} = \frac{3\left(\frac{1}{4}\right)-\frac{5}{2}\left(\frac{1}{2}\right)}{\frac{1}{2}\left(\frac{1}{2}-1\right)} = 2$$

$$c_2 = \left[\frac{z-1}{z}\frac{3z^2-\frac{5}{2}z}{(z-\frac{1}{2})(z-1)}\right]_{z=1} = \frac{3-\frac{5}{2}}{1\left(1-\frac{1}{2}\right)} = 1.$$

Thus

$$X(z) = \frac{2z}{z-\frac{1}{2}} + \frac{z}{z-1},$$

which we can look up in the table to find

$$x[n] = u[n]\left[2\left(\frac{1}{2}\right)^n + 1^n\right] = u[n]\left[\left(\frac{1}{2}\right)^{n-1} + 1\right].$$

Repeated poles: The method explained above needs an additional step when a pole appears more than once in the denominator. Suppose some pole p_i shows up r times, or in other words the denominator contains the expression $(z-p_i)^r$. Then the expansion requires all the terms

$$\frac{c_{i_1}z}{z-p_i} + \frac{c_{i_2}z^2}{(z-p_i)^2} + \ldots + \frac{c_{i_r}z^r}{(z-p_i)^r},$$

which can be iteratively (tediously) solved for as

$$c_{i_r} = \left[\frac{(z-p_i)^r}{z^r}X(z)\right]_{z=p_i},$$

$$c_{i_{(r-1)}} = \left\{\frac{(z-p_i)^{r-1}}{z^{r-1}}\left[X(z) - \frac{c_{i_r}z^r}{(z-p_i)^r}\right]\right\}_{z=p_i},$$

$$c_{i_{(r-2)}} = \left\{\frac{(z-p_i)^{r-2}}{z^{r-2}}\left[X(z) - \frac{c_{i_r}z^r}{(z-p_i)^r} - \frac{c_{i_{(r-1)}}z^{r-1}}{(z-p_i)^{r-1}}\right]\right\}_{z=p_i},$$

and so on. Note that you must simplify the expressions inside the braces before doing the $z = p_i$ substitution; if you don't, and if there are any $(z-p_i)$ denominators, you will find it difficult to evaluate the result correctly.