MAS160: Signals, Systems & Information for Media Technology

Problem Set 3

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Problem 1: Basis Functions

It is possible to use the definition of orthonormality to derive sets of basis functions. What we will use here is a simple version of what is known as the Graham-Schmidt Procedure.

Assume we care about our functions only on the interval $(-1 \leq t \leq 1)$. First we choose some basis function $\phi_0(t)$ such that it satisfies the requirement

$$\int_{-1}^{1} \phi_0^2(t)\,\mathrm{d}t = 1.$$

Then a second member of the basis set, $\phi_1(t)$, must satisfy

$$\int_{-1}^{1} \phi_1^2(t)\,\mathrm{d}t = 1$$

and

$$\int_{-1}^{1} \phi_0(t)\phi_1(t)\,\mathrm{d}t = 0;$$

a third member $\phi_2(t)$ must satisfy three equations, and so forth.

Consider (on the interval from -1 to 1) the basis functions

$$\phi_0(t) = A\ ,$$
$$\phi_1(t) = Bt + C \text{ and}$$
$$\phi_2(t) = Dt^2 + Et + F\ .$$

(a) What are the values of A, B, C, D, E and F needed to make these orthonormal on this interval?

(b) Sketch these three basis functions in the interval $(-1 \leq t \leq 1)$.

(c) What are the coefficients for a series approximation (using $\phi_0(t)$, $\phi_1(t)$, and $\phi_2(t)$) of the function $p(t) = 1 + \cos(\pi t)$ for $-1 \leq t \leq 1$?

These functions, the $\phi_i(t)$'s, are known as Legendre polynomials, and a tremendous amount is known about them. If you're interested you can find out much, much more at mathworld. http://mathworld.wolfram.com/LegendrePolynomial.html

SOLUTION :

(a) *Start by normalising* ϕ_0.

$$\int_{-1}^{1} \phi_0^2(t)\,\mathrm{d}t = 2A^2 \to A = \frac{1}{\sqrt{2}}$$

Then ϕ_1 *be orthogonal to* ϕ_0

$$\int_{-1}^{1} \phi_0(t)\phi_1(t)\,\mathrm{d}t = C\sqrt{2} \to C = 0$$

and normalized

$$\int_{-1}^{1} \phi_1^2(t)\,\mathrm{d}t = \frac{2}{3}B^2 \to B = \sqrt{\frac{3}{2}}\ .$$

Finally ϕ_2 *must be orthogonal to* ϕ_1

$$\int_{-1}^{1} \phi_1(t)\phi_2(t)\,\mathrm{d}t = E\sqrt{2\frac{2}{3}} \to E = 0\ ,$$

and ϕ_2 *must be orthogonal to* ϕ_0

$$\int_{-1}^{1} \phi_0(t)\phi_2(t)\,\mathrm{d}t = \frac{\frac{2\,D}{3} + 2\,F}{\sqrt{2}} = 0 \tag{1}$$

and normalized

$$\int_{-1}^{1} \phi_2^2(t)\,\mathrm{d}t = \frac{2\,D^2}{5} + \frac{4\,D\,F}{3} + 2\,F^2 = 1 \tag{2}$$

Together 1 and 2 can be solved to give

$$D = \pm\frac{3\sqrt{\frac{5}{2}}}{2} \ \textit{and}\ F = \mp\frac{\sqrt{\frac{5}{2}}}{2}$$

Choosing between these two solutions is a matter of convention. Typically the solutions is chosen that makes the coefficient of the highest power postive so

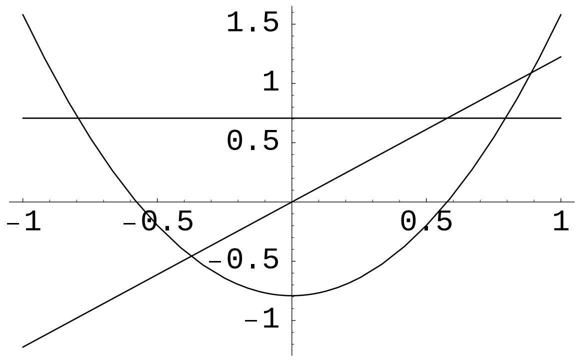
$$D = \frac{3\sqrt{\frac{5}{2}}}{2} \ \textit{and}\ F = -\frac{\sqrt{\frac{5}{2}}}{2}$$

(c) *The coeffients in the expansion are determined from*

$$X_k = \int_{-1}^{1} \phi_k(t)(1 + \cos(\pi t))\,\mathrm{d}t$$

Performing these three integrals gives

$$X_0 = \sqrt{2} \qquad X_1 = 0 \qquad X_2 = -\frac{3\sqrt{10}}{\pi^2}$$



(b)

Figure 1: The First Three Basis Fuctions

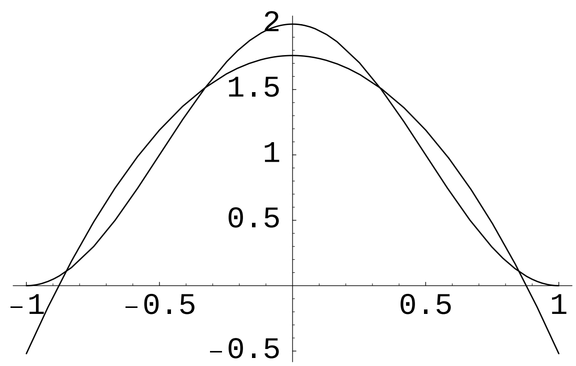


Figure 2: Comparison Between the Series and the Function

Problem 2: AM and Sampling

(*DSP First* 4.6)

SOLUTION :

(a) *See plot of signal and spectrum on next page.*

(b) *The waveform is periodic. Both the carrier and envelope frequencies are multiples of 2000 (see plot of signal and envelope), so the overall period is* $T = \frac{1}{f} = \frac{1}{2000} = 0.005$ *sec.*

(c) *Strictly speaking, the sampling rate,* f_s*, must be more than twice as high as the highest frequency in the signal. Since the highest frequency* $f_c + f_{Delta} = 12000$ *Hz,* $f_s > 24000$ *Hz. However, there are tricks involved with modulated signals that often allow you to sample at lower sampling rates than the strict Nyquist rate and still recover the signal of interest. We will get into these techniques later in the term.*

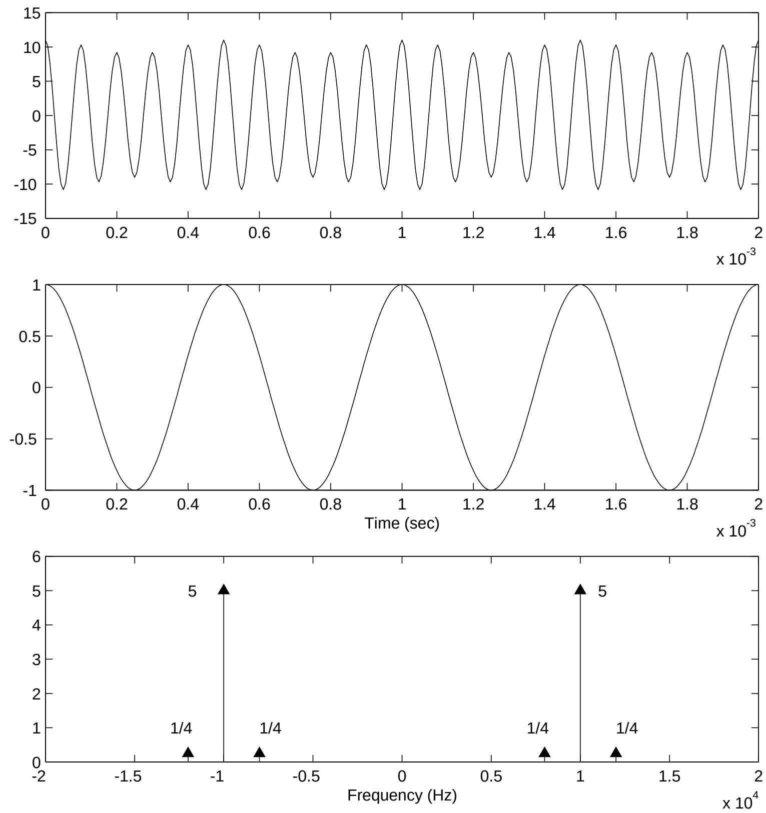


Figure 3: AM signal (top), envelope (middle), and spectrum (bottom). Frequencies indicated in spectrum are $\pm f_c \pm f_{Delta}$, where $f_c = 10000$ Hz and $f_{Delta} = 2000$ Hz.

Problem 3: Frequency, Sampling and Bit Rate

The high-frequency limit of human hearing extends to approximately 20,000 Hz, but studies have shown that intelligible speech requires frequencies only up to 4,000 Hz.

(a) Justify why the sampling rate for an audio Compact Disc (CD) is 44.1 kHz.

(b) What is the Nyquist rate for reliable speech communications? Why do you think people sound different on the phone from in person?

The bit rate of a system can be calculated quite simply as follows:

$$\text{bit rate} = (\text{sampling rate})\ (\text{number of bits per sample})$$

(c) Suppose intelligible speech requires 7 bits per sample. If the phone system is designed to just meet the requirements for speech (which is the case), what is the maximum bit rate allowable over telephone lines? From your result, do you think computer modems (not cable modems, ISDN, or DSL) will get any faster?

(d) CDs use 16 bits per sample. What is the bit rate of music coming off a CD? Is a modem connection fast enough to support streamed CD quality audio?

SOLUTION :

(a) *Since the full range of human hearing extends to 20,000 Hz (20kHz), that is the highest frequency that needs to be represented. To accurately represent those frequencies, we must sample at a rate greater than twice that (40 kHz). The extra 4.1 kHz is to allow some extra room during discrete-to-continuous reconstruction, so that interpolation isn't quite as difficult to implement (remember that the ideal interpolator requires infinte length).*

(b) *Since intelligibility of speech requires frequencies only up to 4000 Hz, the Nyquist rate is twice that, or 8000 Hz (8 kHz). The human voice does contain frequencies about 4 kHz, it's just that they're not necessary to understand the words being spoken. Since the phone only carries the frequencies necessary to understand speech, there's a whole range of frequencies in speech which are not transmitted over the phone, hence the difference in a person's voice over the phone vs. in person.*

(c) *Since the phone system essentially "samples" up to 8 kHz (the Nyquist rate required for intelligible speech), and 7 bits per sample are also required, we get*

$$\textit{bit rate} = 8 \ \textit{k samples/sec} \times 7 \ \textit{bits/sample} = 56 \ \textit{kbps}$$

Since this is the bit rate achieved by current modems, it is unlikely that conventional computer modems will get faster. (Granted, this argument is an oversimplification of the problem, but the basic principles are true.)

(d) *CD bit rate = 44.1 k samples/sec × 16 bits/sample = 705.6 kbps*
This is much ($> 10\times$) *higher than the bit rate of a modem, so it is unlikely that a modem connection is fast enough to support CD quality audio.*

Problem 4: Non-ideal D-to-C Conversion

(*DSP First* 4.8)

SOLUTION :

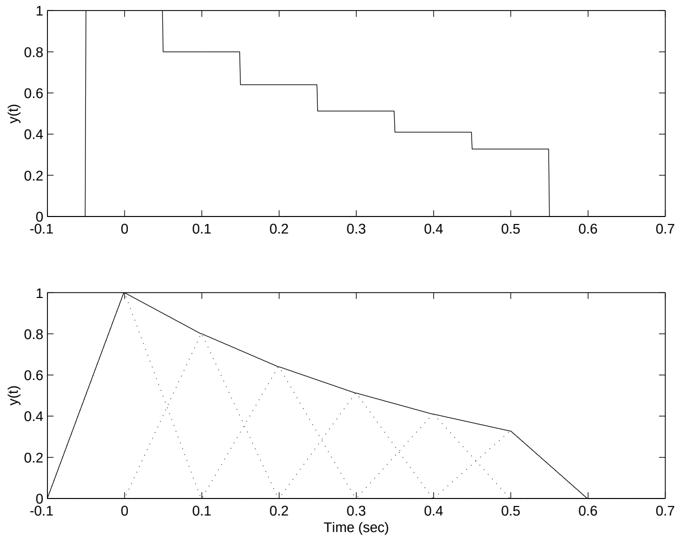


Figure 4: Discrete-to-continuous conversion using different p(t) functions: (a) (top), (b) (bottom).

Problem 5: Representing Irrational Frequencies (for MAS 510)

Later in this course we will describe how if a signal is periodic in time, then it is discrete in frequency, and vice-versa. At first glance, Fig 3.17 seems to violate this statement; however, if you look closer, it does not. Let's explore what is going on.

Consider the following signals

$$x(t) = 2\cos(10\pi\sqrt{8}t) + 3\cos(30\pi\sqrt{27}t) \tag{3}$$

$$y(t) = 2\cos(10\pi t) + 3\cos(30\pi t) \tag{4}$$

(a) Plot the two signals in the time domain on the same page.

(b) Plot the two signals in the frequency domain using the `stem` function in MATLAB.

Is x(t) really discrete in the frequency domain? What is the computer's approximation of the irrational frequencies?

SOLUTION :

(a) *Plot of both signals*

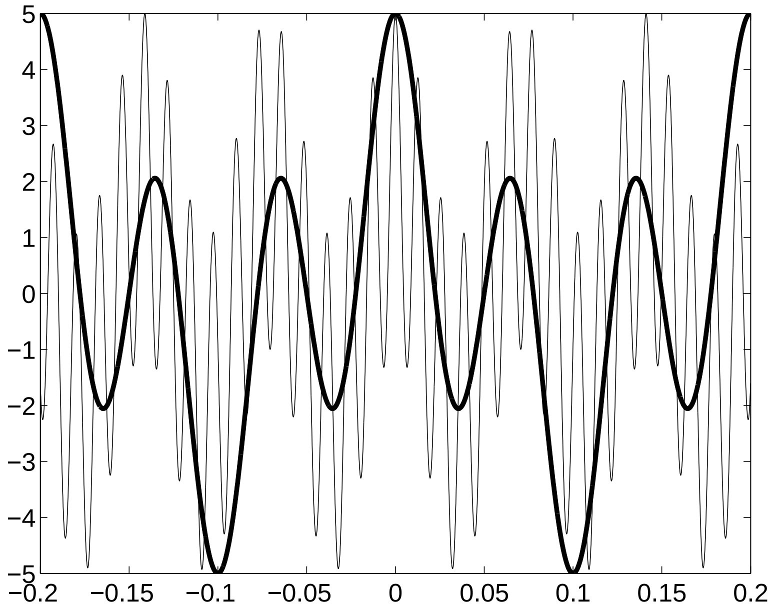


Figure 5: An Overlay of the $x(t)$ and $y(t)$ ($y(t)$ is bold)

(b) *Looking at the full spectrum it seems that the spectrum containg the irrational requencies is slightly flared near the spikes. If you zoom into the spectra and view them with the stem plot the difference is apparent. The spectrum from the signals containg only integer frequencies has contributions just at those frequencies. The spectrum from the signal containing irrational frequencies actually has power in a small range around the expected points. As more and more of the signal is analysed, this range will shrink until (in the infinite sample limit) only single spikes are left.*

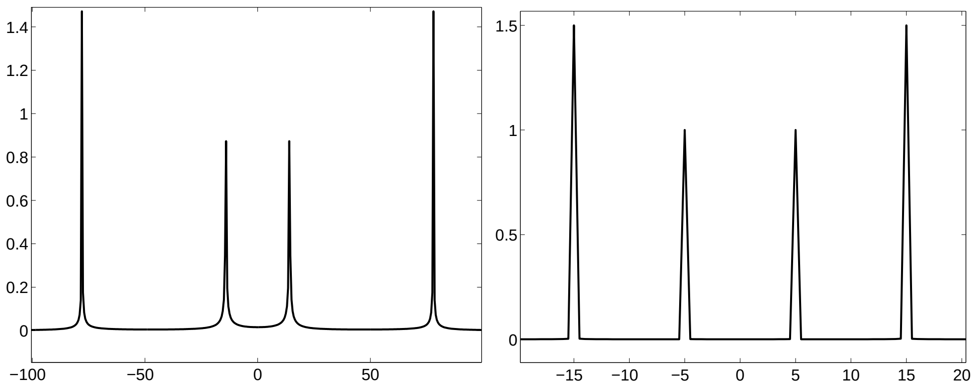


Figure 6: The Spectra of $x(t)$ and $y(t)$

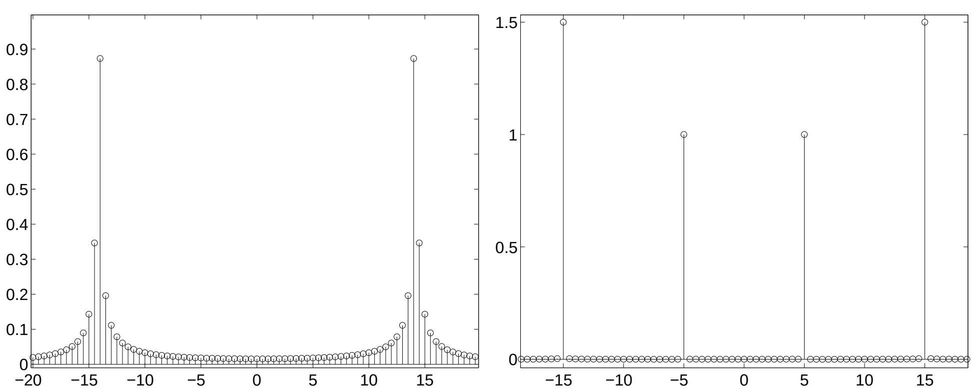


Figure 7: Zooming into the Spectrum of $x(t)$