

MAS160: Signals, Systems & Information for Media Technology

Problem Set 5

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Problem 1: Unit-step and running average (DSP First 5.5)

SOLUTION :

(a)

(b)

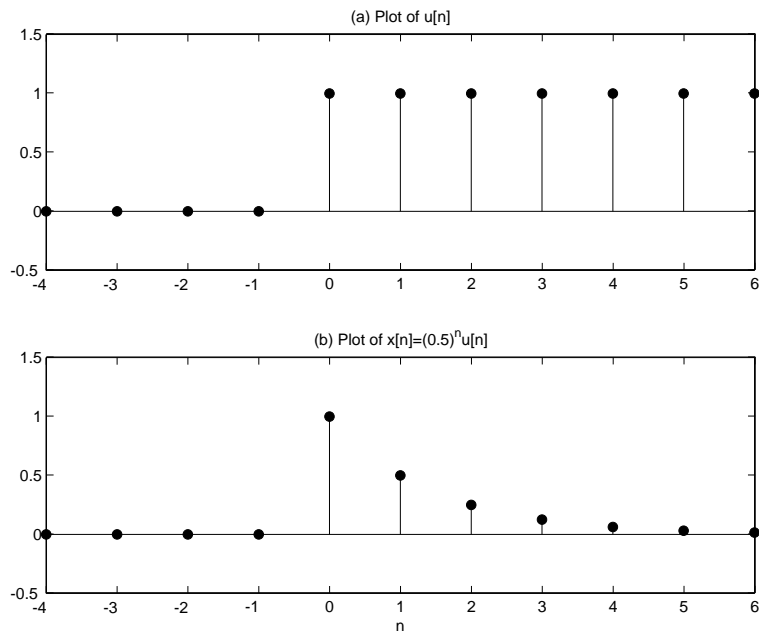


Figure 1: Unit Step and Exponential Decay

(c)

$$y[n] = \frac{1}{4} \sum_{k=0}^3 (0.5)^{n-k} u[n-k]$$

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$y[n]$	0	0	0	0	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{15}{32}$	$\frac{15}{64}$	$\frac{15}{128}$	$\frac{15}{256}$	$\frac{15}{512}$	$\frac{15}{1024}$	$\frac{15}{2048}$	$\frac{15}{4096}$

(d)

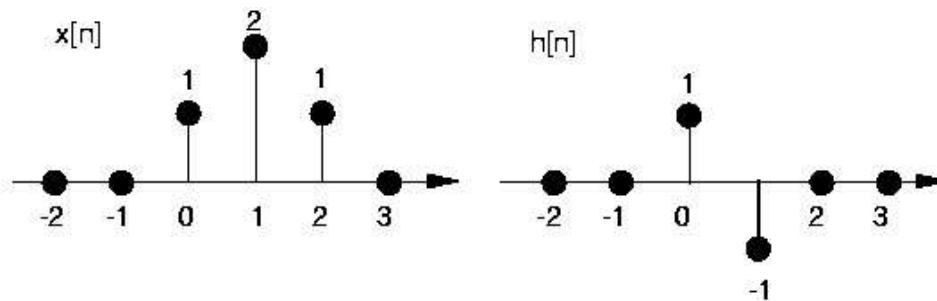
$$y[n] = \frac{1}{L} \sum_{k=\max\{0, n-L+1\}}^n a^k = \frac{a^{\max\{0, n-L+1\}} - a^{n+1}}{L(1-a)}$$

The max operator simply means that we take the argument of largest value. So in this case, it is either 0 or $n - L + 1$, whichever is larger (which will depend on the values of n and L).

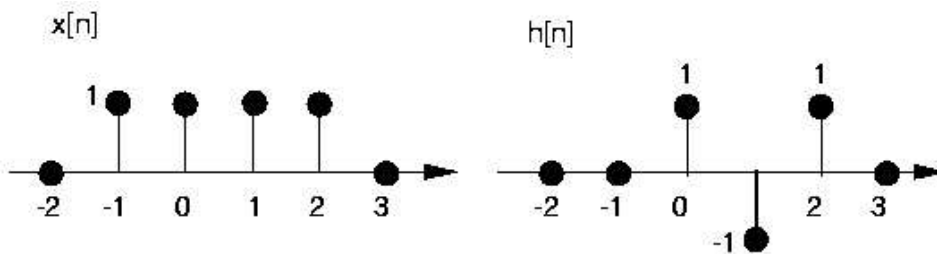
Problem 2: Convolution

For each of the following sets of signals, compute their convolution (1) graphically by hand, (2) with MATLAB (you may use the `conv` function), and (3) by expressing the signals in terms of $\delta[n]$ and computing the convolution sum. In MATLAB, plot your results with `stem`, but be sure to fix the n -axis appropriately (use `stem(n,y)` where n is a vector of the appropriate range).

(a)



(b)



For each of the following of signals, compute their convolution with $x[n] = \cos(2\pi(\frac{1}{16})n)$ using MATLAB (you may use the `conv` function). Use `stem` to plot your result over the range $[0:99]$, assuming the sinusoid exists for all time. Compare each convolution with $x[n]$.

(c) $h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n - 1]$

(d) $h[n] = \delta[n] - \delta[n - 1]$

SOLUTION :

For the “graphically by hand” portions of this problem, I intended for you to break down the $x[n]$ into separate scaled and shifted impulses, convolve them separately, and then sum the results. Some people interpreted this to mean to solve the convolution by hand in tabular form (which really amounts to the same thing), which I also accepted.

(a) The graphical solution is demonstrated in the figure below. Obviously, the MATLAB solution is identical to the final summation.

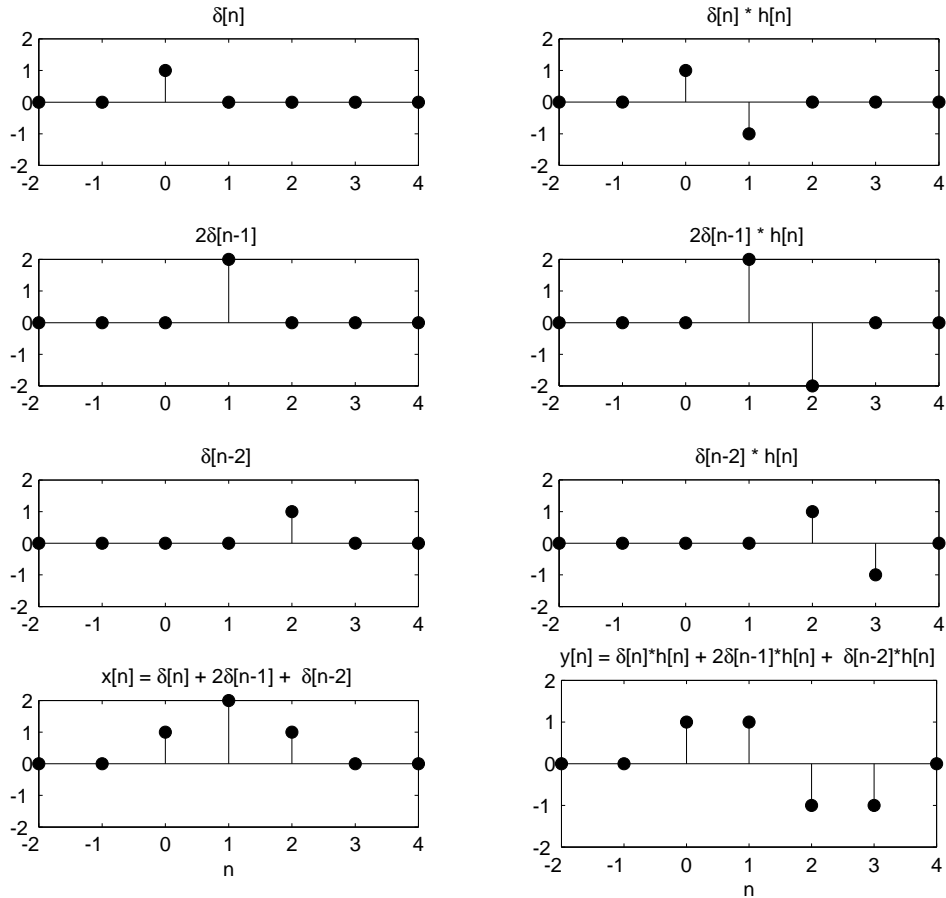


Figure 2: Pointwise Convolution for (a)

Here is the solution expressed in tabular form, this time breaking down $h[n]$ into separate scaled and shifted impulses:

n	-2	-1	0	1	2	3	4
$x[n]$	0	0	1	2	1	0	
$h[n]$	0	0	1	-1	0	0	
$h[0]x[n-0]$	0	0	1	2	1	0	
$h[1]x[n-1]$		0	0	-1	-2	-1	0
$x[n] * h[n]$	0	0	1	1	-1	-1	0

Below is the convolution sum performed by expressing the signals in terms of $\delta[n]$:

$$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$h[n] = \delta[n] - \delta[n-1]$$

$$y[0] = \sum_{k=0}^2 x[k]h[0-k] = x[0]h[0] = (1)(1) = 1$$

$$y[1] = \sum_{k=0}^2 x[k]h[1-k] = x[0]h[1] + x[1]h[0] = (1)(-1) + (2)(1) = 1$$

$$y[2] = \sum_{k=0}^2 x[k]h[2-k] = x[0]h[2] + x[1]h[1] + x[2]h[0] = (1)(0) + (2)(-1) + (1)(1) = -1$$

$$y[3] = \sum_{k=0}^2 x[k]h[3-k] = x[0]h[3] + x[1]h[2] + x[2]h[1] = (1)(0) + (2)(0) + (1)(-1) = -1$$

$$y[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$$

(b) Again, the graphical solution is demonstrated in the figure below.

The solution expressed in tabular form:

n	-2	-1	0	1	2	3	4	5
$x[n]$	0	1	1	1	1	0		
$h[n]$	0	0	1	-1	1	0		
$h[0]x[n-0]$	0	1	1	1	1	0		
$h[1]x[n-1]$		0	-1	-1	-1	-1	0	
$h[2]x[n-2]$			0	1	1	1	1	0
$x[n] * h[n]$	0	1	0	1	1	0	1	0

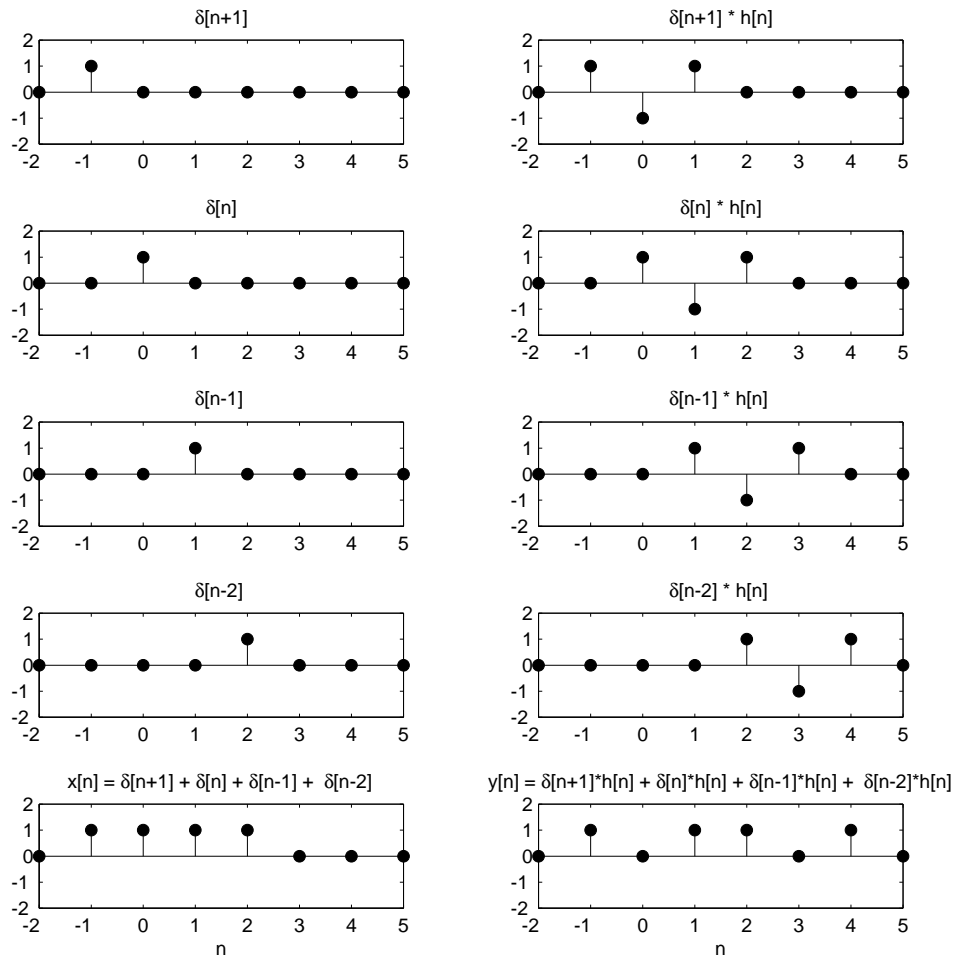


Figure 3: Pointwise Convolution for (b)

The convolution sum is achieved mathematically as follows:

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]$$

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2]$$

$$y[-1] = \sum_{k=-1}^2 x[k]h[-1-k] = x[-1]h[0] = (1)(1) = 1$$

$$y[0] = \sum_{k=-1}^2 x[k]h[-k] = x[-1]h[1] + x[0]h[0] = (1)(-1) + (1)(1) = 0$$

$$y[1] = \sum_{k=-1}^2 x[k]h[1-k] = x[-1]h[2] + x[0]h[1] + x[1]h[0] = (1)(1) + (1)(-1) + (1)(1) = 1$$

$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] = (1)(1) + (1)(-1) + (1)(1) = 1$$

$$y[3] = x[1]h[2] + x[2]h[1] = (1)(1) + (1)(-1) = 0$$

$$y[4] = x[2]h[2] = (1)(1) = 1$$

$$y[n] = \delta[n+1] + \delta[n-1] + \delta[n-2] \quad \text{PS } 5-5$$

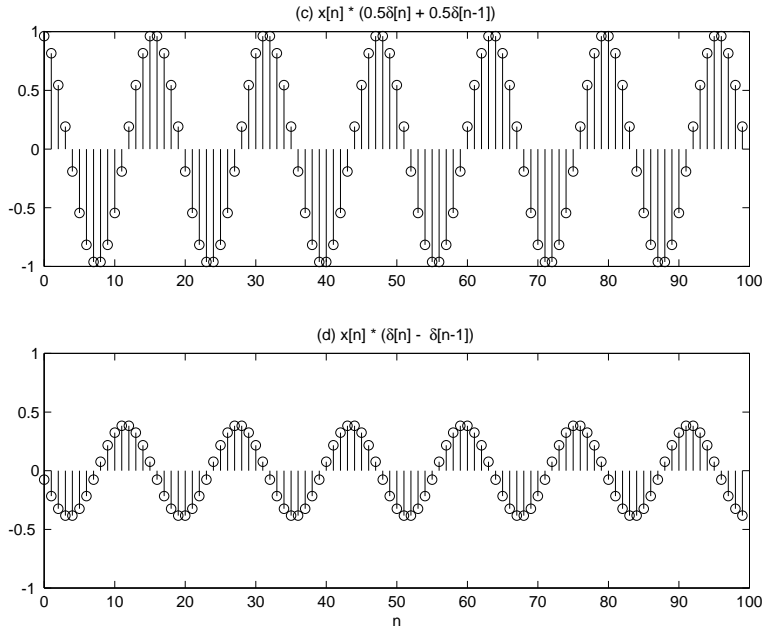


Figure 4: (c) A Low Pass Filter (d) A High Pass Filter

- (c)
 (d) The filter in part (c) is an averaging (low-pass) filter. Since $x[n]$ is of fairly low frequency, the signal doesn't change much. In part (d), the filter is a difference (generally high-pass) filter, which does affect the amplitude substantially.

Problem 3: Time-domain response of FIR filters (DSP First 5.6)

- (a) From the given plot, we get: $\frac{b_0}{3} \mid \frac{b_1}{7} \mid \frac{b_2}{13} \mid \frac{b_3}{9} \mid \frac{b_4}{5}$
 (b)

$$\begin{aligned}
 y[n] &= \sum_{k=0}^M b_k x[n - k] \\
 &= 13x[n] - 13x[n - 1] + 13x[n - 2]
 \end{aligned}$$

But since

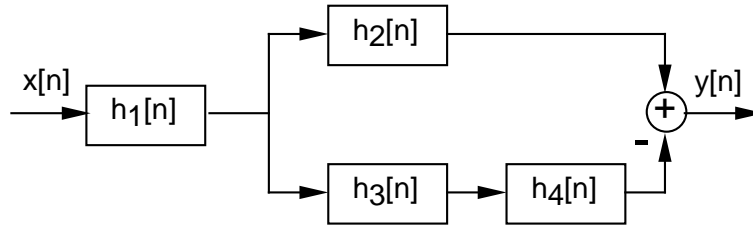
$$x[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ 1, & \text{for } n \text{ odd} \end{cases}$$

We see that $x[n] = x[n - 2]$. Therefore:

$$\begin{aligned}
 y[n] &= 26x[n] - 13x[n - 1] \\
 &= \begin{cases} -13, & \text{for } n \text{ even} \\ 26, & \text{for } n \text{ odd} \end{cases}
 \end{aligned}$$

Problem 4: LTI Systems

Consider the interconnection of LTI systems as shown below.



- (a) Express the overall impulse response, $h[n]$, in terms of $h_1[n]$, $h_2[n]$, $h_3[n]$ and $h_4[n]$.
(b) Determine $h[n]$ when

$$\begin{aligned}h_1[n] &= \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{2} \right\} \\h_2[n] &= h_3[n] = (n+1)u[n] \\h_4[n] &= \delta[n-2]\end{aligned}$$

SOLUTION :

- (a) *The combined impulse response of a cascade of two LTI systems is simply the convolution of the two impulse responses. The impulse response of an addition (or subtraction) of two LTI systems is simply the sum (or difference) of the two impulse responses. Thus, for the overall system impulse response, we obtain:*

$$h[n] = h_1[n] * \{h_2[n] - (h_3[n] * h_4[n])\}$$

(b) Proceeding in steps, we get:

$$\begin{aligned}
 h_3[n] * h_4[n] &= ((n+1)u[n]) * \delta[n-2] \\
 &= \sum_{k=-\infty}^{\infty} \delta[k-2](n-k+1)u[n-k] \quad (\text{only nonzero when } k=2) \\
 &= (n-1)u[n-2]
 \end{aligned}$$

$$\begin{aligned}
 h_5[n] &\triangleq h_2[n] - (h_3[n] * h_4[n]) \\
 &= (n+1)u[n] - (n-1)u[n-2] \\
 &= \begin{cases} 0, & n > 0 \\ 1, & n = 0 \\ 2, & n > 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 h[n] &= h_1[n] * h_5[n] \\
 &= \frac{1}{2}h_5[n] + \frac{1}{4}h_5[n] + \frac{1}{2}h_5[n] \\
 &= \begin{cases} 0, & n < 0 \\ \frac{1}{2}, & n = 0 \\ \frac{5}{4}, & n = 1 \\ 2, & n = 2 \\ \frac{5}{2}, & n > 2 \end{cases}
 \end{aligned}$$

Problem 5: Block Diagrams (*DSP First 5.9*)

SOLUTION :

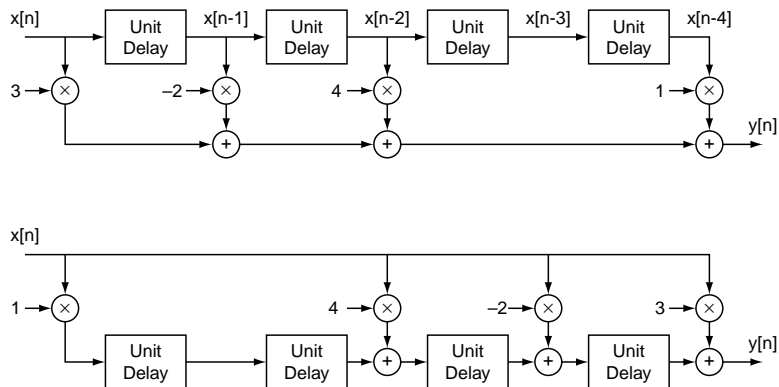


Figure 5: System Diagram in Direct Form (above) and Transposed Direct Form (below)

Problem 6: MAS.510 Additional Problem

It is possible to determine the impulse response for a LTI system using a system of equations, given enough information about the system. For example, if we know that the system is FIR and has no delay and that $y[0] = 1$ if $x[n] = \delta[n]$, then

$$\begin{aligned}y[n] &= ax[n] \\y[0] &= ax[0] \\1 &= a * 1 \\a &= 1\end{aligned}$$

Using systems of equations, compute the impulse response given the following system descriptions and input-output pairs

- (a) FIR and single delay, $x[n] = \delta[n]$, $y[0] = 2$, $y[1] = -2$
- (b) FIR and double delay, $x[n] = \delta[n]$, $y[0] = 3$, $y[1] = -4$, $y[2] = 3/2$
- (c) FIR and double delay, $x[n] = 4\delta[n] + \delta[n - 1]$, $y[0] = 2$, $y[1] = 2$, $y[2] = -1$
- (d) Calculate $y[3]$ for each of the preceding systems.

SOLUTION :

- (a) *Since the input $x[n]$ is an impulse, we could simply get the impulse response $h[n]$ by inspection. But just to show the math we'll do this one the long way:*

$$\begin{aligned}y[n] &= ax[n] + bx[n - 1] \\y[0] &= ax[0] + bx[-1] \\2 &= a(1) + b(0) \\a &= 2\end{aligned}$$

$$\begin{aligned}y[1] &= 2x[1] + bx[0] \\-2 &= 2(0) + b(1) \\b &= -2\end{aligned}$$

$$\begin{aligned}y[n] &= 2x[n] - 2x[n - 1] \\h[n] &= 2\delta[n] - 2\delta[n - 1]\end{aligned}$$

- (b) *This one also has an impulse input, so this time we can simply obtain the impulse response by inspection:*

$$h[n] = 3\delta[n] - 4\delta[n - 1] + \frac{3}{2}\delta[n - 2]$$

(c) Since the input is not just an impulse, we actually have to perform a deconvolution using a system of equations to get our answer.

$$x[n] = 4\delta[n] + \delta[n - 1]$$

$$= \begin{cases} 4, & n = 0 \\ 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = ax[n] + bx[n - 1] + cx[n - 2]$$

$$y[0] = ax[0] + bx[-1] + cx[-2]$$

$$2 = a(4) + b(0) + c(0)$$

$$a = \frac{1}{2}$$

$$y[1] = ax[1] + bx[0] + cx[-1]$$

$$2 = \frac{1}{2}(1) + b(4) + c(0)$$

$$b = \frac{3}{8}$$

$$y[2] = ax[2] + bx[1] + cx[0]$$

$$-1 = \frac{1}{2}(0) + \frac{3}{8}(1) + c(4)$$

$$c = -\frac{11}{32}$$

$$y[n] = \frac{1}{2}x[n] + \frac{3}{8}x[n - 1] - \frac{11}{32}x[n - 2]$$

$$h[n] = \frac{1}{2}\delta[n] + \frac{3}{8}\delta[n - 1] - \frac{11}{32}\delta[n - 2]$$

(d) Using the coefficients above, it is simple to calculate $y[3]$:

$$y[3] = 2x[3] - 2x[2] = 2(0) - 2(0) = 0 \tag{a}$$

$$y[3] = 3x[3] - 4x[2] + \frac{3}{2}x[1] = 3(0) - 4(0) + \frac{3}{2}(0) = 0 \tag{b}$$

$$y[3] = \frac{1}{2}x[3] + \frac{3}{8}x[2] - \frac{11}{32}x[1] = \frac{1}{2}(0) + \frac{3}{8}(0) - \frac{11}{32}(1) = \frac{11}{32} \tag{c}$$