MAS160: Signals, Systems & Information for Media Technology

Problem Set 6

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Problem 1: Frequency response of FIR filters (*DSP First* 6.4)

SOLUTION :

(a)

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} \\ &= 2e^{-j\hat{\omega}0} - 3e^{-j\hat{\omega}1} + 2e^{-j\hat{\omega}2} \\ &= 2 - 3e^{-j\hat{\omega}} + 2e^{-j\hat{\omega}2} \\ &= e^{-j\hat{\omega}}\left(2e^{-j\hat{\omega}} - 3 + 2e^{-j\hat{\omega}}\right) \\ &= e^{-j\hat{\omega}}\left(-3 + 4\cos(\hat{\omega})\right)\end{aligned}$$

$$\begin{aligned}|\mathcal{H}(\hat{\omega})| &= |-3 + 4\cos(\hat{\omega})| \\ \angle\mathcal{H}(\hat{\omega}) &= -\hat{\omega}\end{aligned}$$

(b) $\mathcal{H}(\hat{\omega})$ *has a period of* 2π.

(c) See figure 1

(d)

$$\begin{aligned}-3 + 4\cos(\hat{\omega}) &= 0 \\ \cos(\hat{\omega}) &= \frac{3}{4} \\ \hat{\omega} &= \cos^{-1}\left(\frac{3}{4}\right) \\ &\approx \pm 0.7227 \pm 2\pi k \text{ radians, } k \text{ integer}\end{aligned}$$

(e) Since $x[n] = \sin(\frac{\pi}{13}n)$, *we need only evaluate the frequency response* $\mathcal{H}(\hat{\omega})$ *at* $\hat{\omega} = \frac{\pi}{13}$:

$$\begin{aligned}\mathcal{H}\left(\frac{\pi}{13}\right) &= e^{-j\pi/13}\left(-3 + 4\cos\left(\frac{\pi}{13}\right)\right) \\ &\approx 0.8838e^{-j\pi/13}\end{aligned}$$

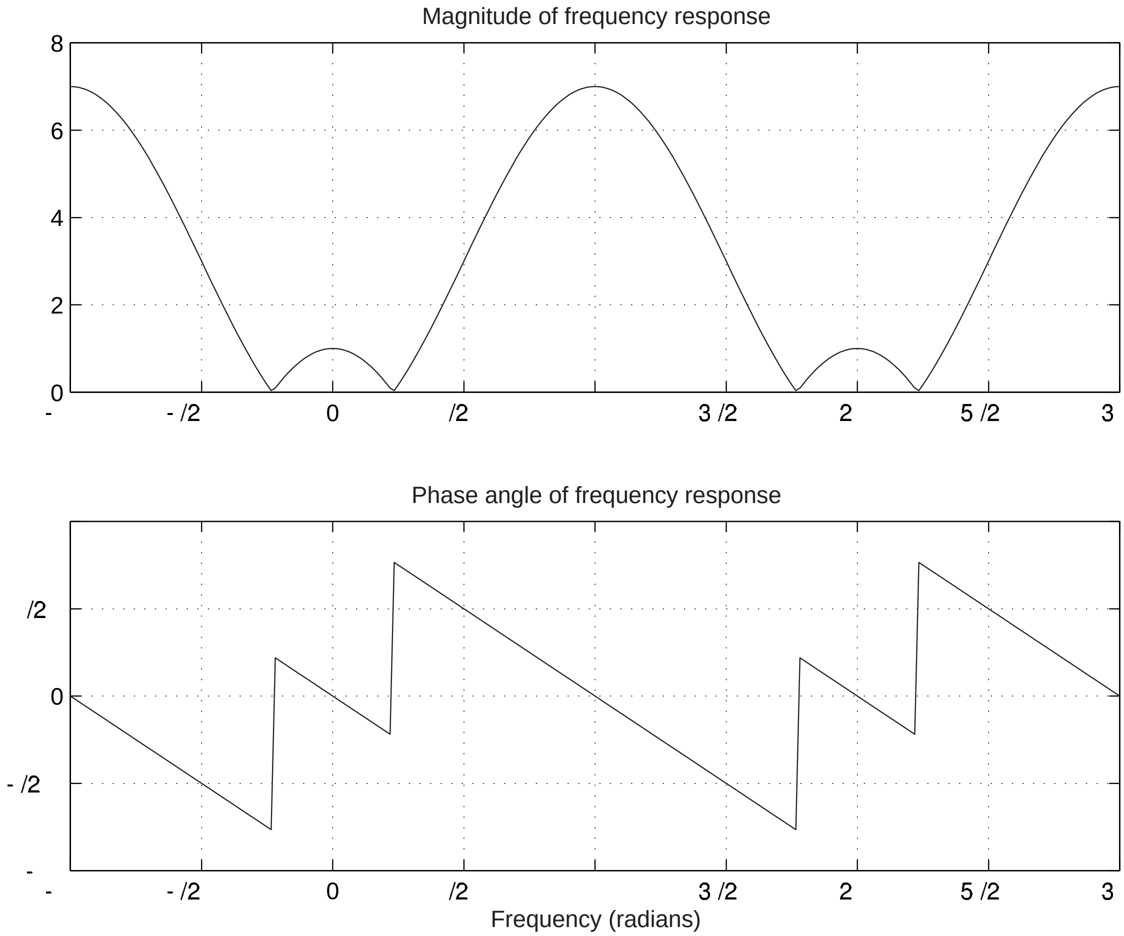

Figure 1: Frequency Response of $y[n] = 2x[n] - 3x[n-1] + 2x[n-2]$

Therefore, the magnitude is $|\mathcal{H}(\frac{\pi}{13})| \approx 0.8838|$ *and the phase is* $\angle|\mathcal{H}(\frac{\pi}{13})| = -\frac{\pi}{13}$*, so the output* $y[n]$ *is:*

$$\begin{aligned}
y[n] &= 0.8838 \sin\left(\frac{\pi}{13}n - \frac{\pi}{13}\right) \\
&= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{\pi}{13} - \frac{\pi}{2}\right) \\
&= 0.8838 \cos\left(\frac{\pi}{13}(n-1) - \frac{\pi}{2}\right) \\
&= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{15\pi}{26}\right)
\end{aligned}$$

Problem 2: Simple sound filtering

Using our old friend the `sumcos` function, create a sound with a fundamental frequency of 440 Hz, with 12 harmonics of equal amplitude and zero phase, and using the following parameters:

```
fs = 11025;          % Sets sampling rate to 11025 Hz
f = 440*[1:12];      % Creates frequency vector of 12 harmonics of 440 Hz
X = ones(1,12);      % Creates amplitudes of 1
dur = 1;             % Sets duration to be 1 sec
```

Use MATLAB to perform the following tasks:

(a) Create a three-point averaging FIR filter and plot the frequency response (magnitude and phase) of this filter using `freqz`. What is this filter supposed to do? Filter the sound you created above with this filter. How does it compare to the original sound?

(b) Create a two-point first difference FIR filter and plot the frequency response (magnitude and phase) of this filter using `freqz`. What is this filter supposed to do? Filter the original sound you created above using this new filter. How does it compare to the original sound?

SOLUTION :

(a) *This is a low-pass filter designed to filter out high frequencies. You should hear it attenuate the higher frequencies in your sound, much like turning down the treble control on your stereo.*

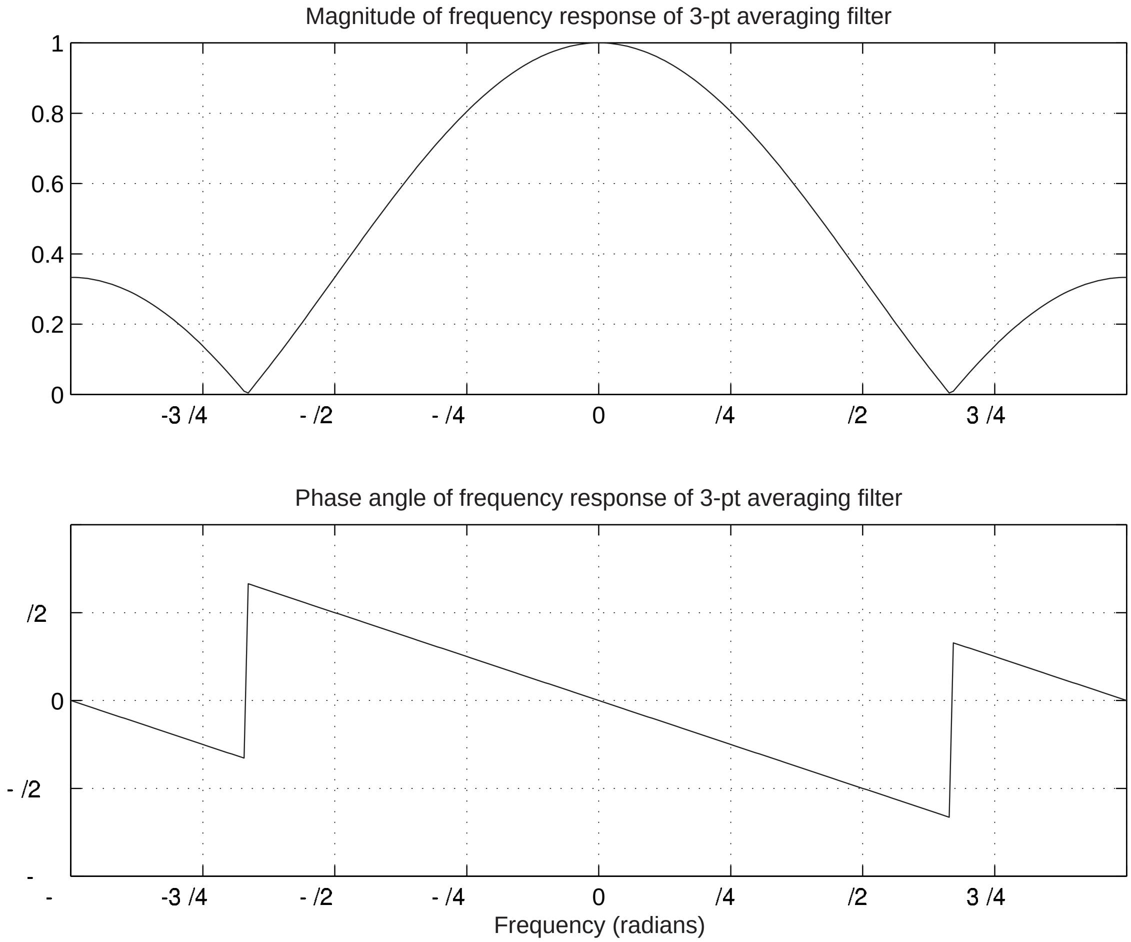

Figure 2: Averaging Filter

(b) *This is a high-pass filter designed to filter out low frequencies. You should hear it attenuate the lower frequencies in your sound, much like turning down the bass control on your stereo.*

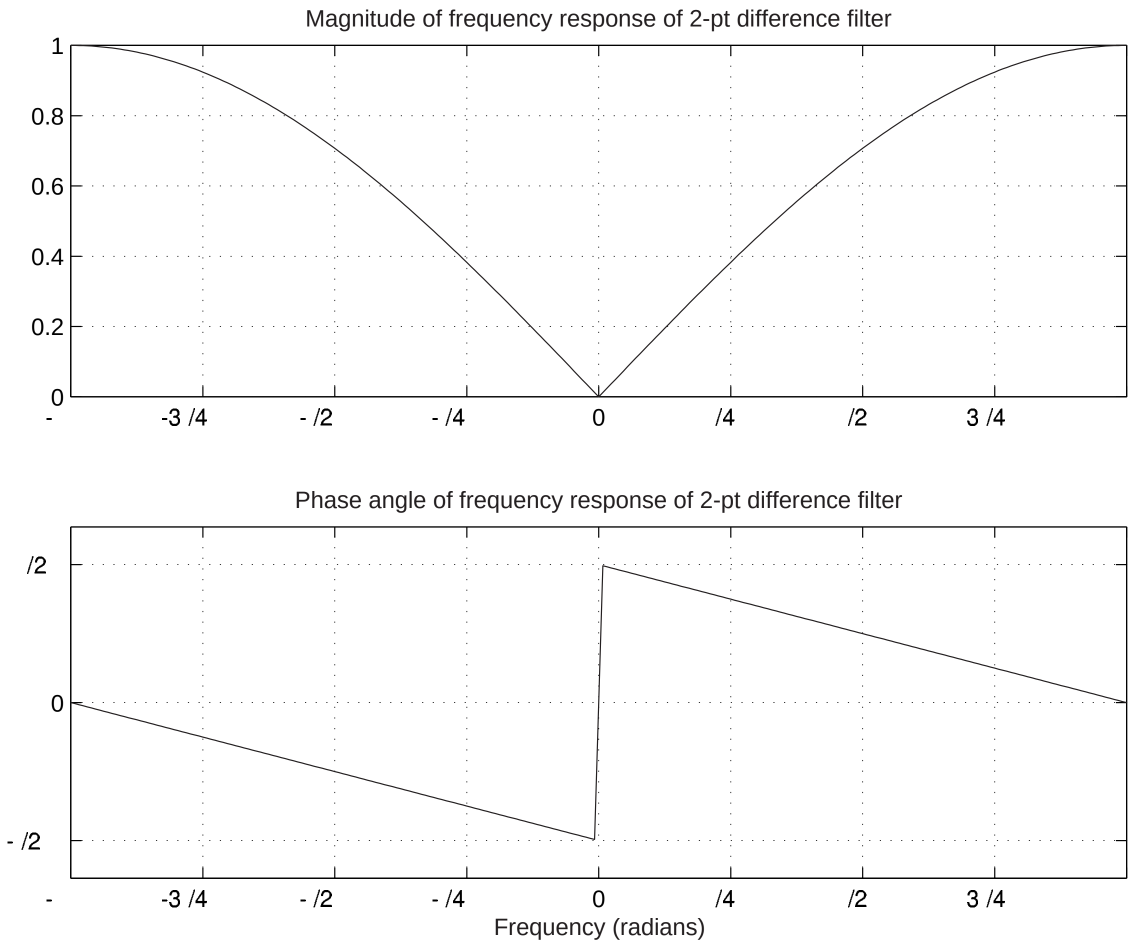

Figure 3: Differencing Filter

Problem 3: Return of the Labs: *DSP First* Lab 5

Items to be turned in:

(a) Plots and answers to questions specified in C.5.2.1.
(b) Plots and answers to questions specified in C.5.3.2.
(c) Demonstrate linearity and time-invariance of filter (C.5.3.3 and C.5.3.4).
(d) Plots and answers to questions specified in C.5.3.5.

SOLUTION :

(a) **C.5.2.1 Frequency Response of the 3-Point Averager**

(1) The impulse response of a three-point averager is simply:

$$h[k] = \frac{1}{3}\delta[k] + \frac{1}{3}\delta[k-1] + \frac{1}{3}\delta[k-2]$$

The frequency response can be calculated as follows:

$$\begin{aligned}
\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} \\
&= \frac{1}{3}e^{-j\hat{\omega}0} + \frac{1}{3}e^{-j\hat{\omega}1} + \frac{1}{3}e^{-j\hat{\omega}2} \\
&= \frac{1}{3} + \frac{1}{3}e^{-j\hat{\omega}} + \frac{1}{3}e^{-j\hat{\omega}2} \\
&= \frac{1}{3}e^{-j\hat{\omega}}\left(e^{-j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right) \\
&= \frac{2\cos(\hat{\omega}) + 1}{3}e^{-j\hat{\omega}}
\end{aligned}$$

(2) The following code computes this frequency response directly in MATLAB*:*

```
function 3ptlowpass()

w = -pi:pi/128:pi;
h = freqz([1/3 1/3 1/3],1,w);

subplot(2,1,1)
plot(w,abs(h))
axis([-pi pi 0 1])
set(gca,'XTick',[-pi:pi/4:pi]);
set(gca,'XTickLabel','|-3/4|-/2|-/4|0|/4|/2|3/4|');
%set(get(gca,'XTickLabel'))
grid
title('Magnitude of frequency response of 3-pt averaging filter')

subplot(2,1,2)
plot(w,angle(h))
axis([-pi pi -pi pi])
set(gca,'XTick',[-pi:pi/4:pi]);
set(gca,'XTickLabel','-|-3/4|-/2|-/4|0|/4|/2|3/4|');
set(gca,'YTick',[-pi:pi/2:pi]);
set(gca,'YTickLabel','-|-/2|0|/2|');
grid
title('Phase angle of frequency response of 3-pt averaging filter')
xlabel('Frequency (radians)')
```

The following plots are generated:

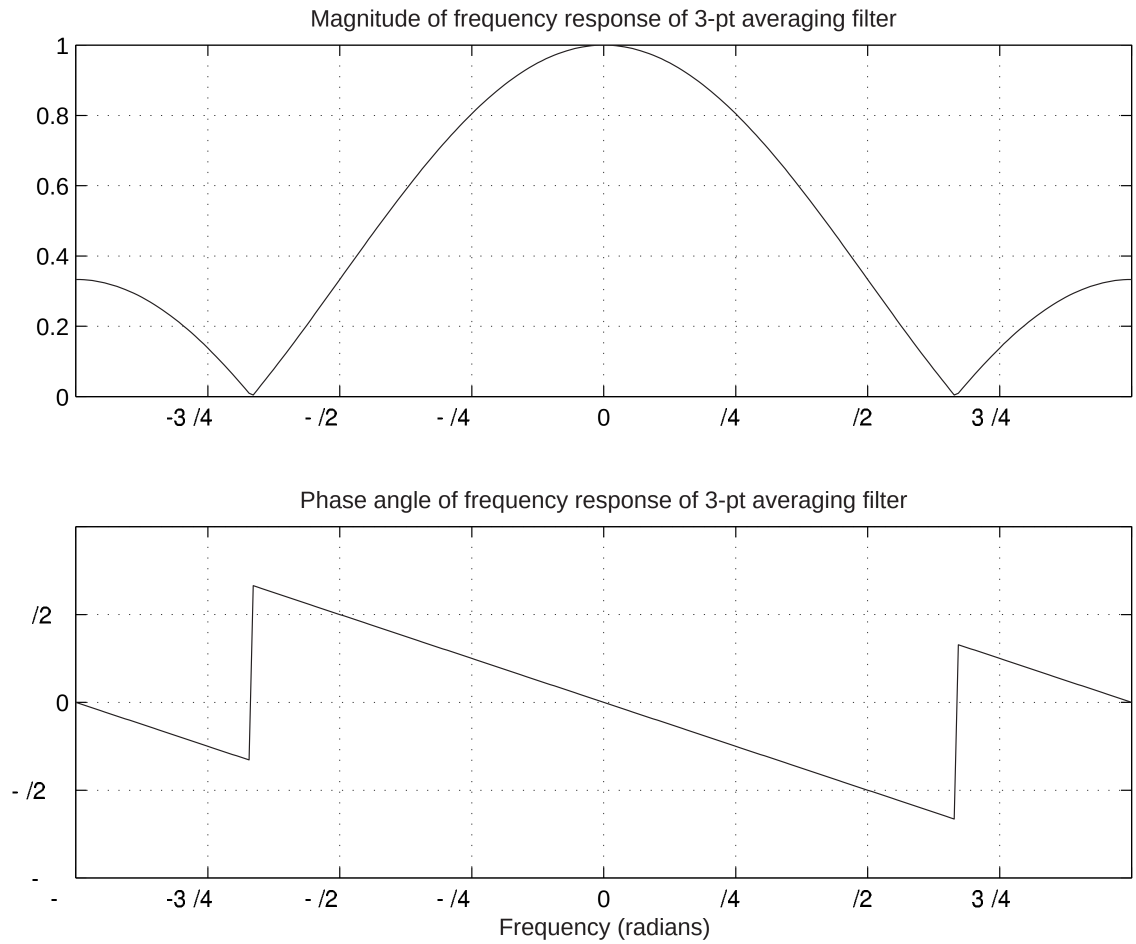

Figure 4: Averaging Filter (again)

(3) The magnitude and phase plots generated computing $H(\hat{\omega})$ directly and using `freqz` *to evaluate the response are identical.*

(b) **C.5.3.2 First-Difference Filter**

(1) $x[n]$ *and* $y[n]$ *are not the same length because the filtering is performed via convolution, resulting in a signal length that is the sum of the individual signal lengths minus one* $(N + M - 1)$.

(2)

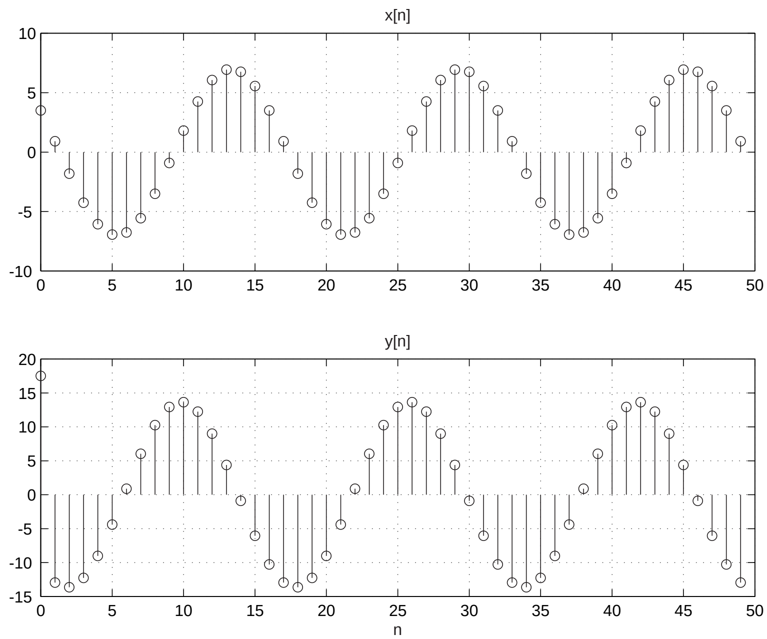

Figure 5: First Order Differencing (Time Domain)

(3) From the plot, it appears that the amplitude of $x[n] = 7$. *The phase is approximately 3 samples which gives us* $3(0.125\pi) \approx \frac{\pi}{3}$.

(4) The filter takes the difference between adjacent samples. Since $x[n]$ *is zero for* $n < 0$, $y[0]$ *is the result of a much greater difference than other values of* $y[n]$.

(5) The frequency is remains the same, $\hat{\omega} = 0.125\pi$. *The amplitude is approximately 14, and the phase is approximately 6 samples, which gives us* $6(0.125\pi) = \frac{3}{4}\pi$.

(6) At the input frequency $\hat{\omega} = 0.125\pi$, *the filter seems to do the following:*

$$Amplitude \approx \frac{14}{7} = 2$$
$$Phase \approx \frac{3\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$$

(7)

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= 5e^{-j\hat{\omega}0} - 5e^{-j\hat{\omega}1} \\ &= 5e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) \\ &= 5e^{-j\hat{\omega}/2}2j\sin(\hat{\omega}/2) \\ &= 10e^{-j(\hat{\omega}-\pi)/2}\sin(\hat{\omega}/2) \\ |\mathcal{H}(0.125\pi)| &= 10\sin(0.125\pi/2) \\ &\approx 1.951 \\ \angle\mathcal{H}(0.125\pi) &= -(0.125\pi - \pi)/2 \\ &= \frac{7}{16}\pi\end{aligned}$$

(c) **C.5.3.3 Linearity and time invariance of the Filter**

(1)

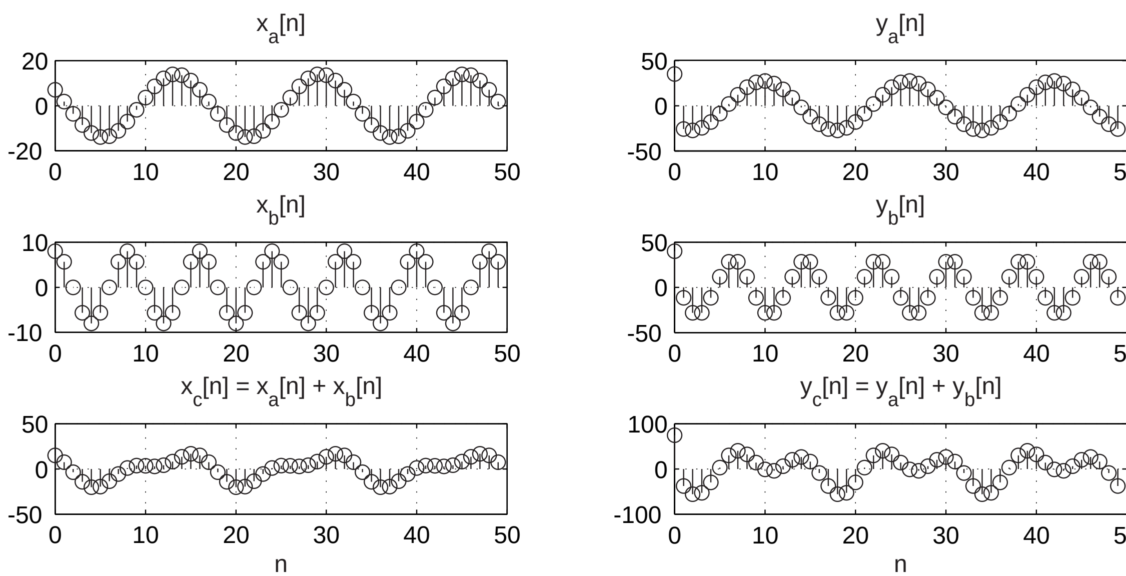

Figure 6: Checking Linearity

From the plot, the amplitude of $x_a[n]$ is approximately 14 and the amplitude of $y_a[n]$ is approximately 28. The phase of $x_a[n]$ is $\frac{\pi}{3}$ and the phase of $y_a[n]$ is about 6 samples, which is $6(0.125\pi) = \frac{3}{4}\pi$. Thus relative change in amplitude is still approximately 2, and the relative change in phase is still $\frac{5}{12}\pi \approx \frac{7}{16}\pi$.

(2) *The amplitude of $x_b[n]$ is 8 and its phase is 0. The amplitude of $y_b[n]$ is approximately 32 and its phase is about 1.5 samples, which is $1.5(0.25\pi) = \frac{3}{8}\pi$. Thus the relative change in amplitude is 4, and the relative change in phase is $\frac{3}{8}\pi$.*

(3) *$y_c[n]$ is identical to $y_a[n] + y_b[n]$, demonstrating that the filter is indeed linear.*

(4) It is clear from the plot that $y[n]$ must also be shifted to the right by 3 samples to be equal to $y_s[n]$, which demonstrates that the filter is time-invariant.

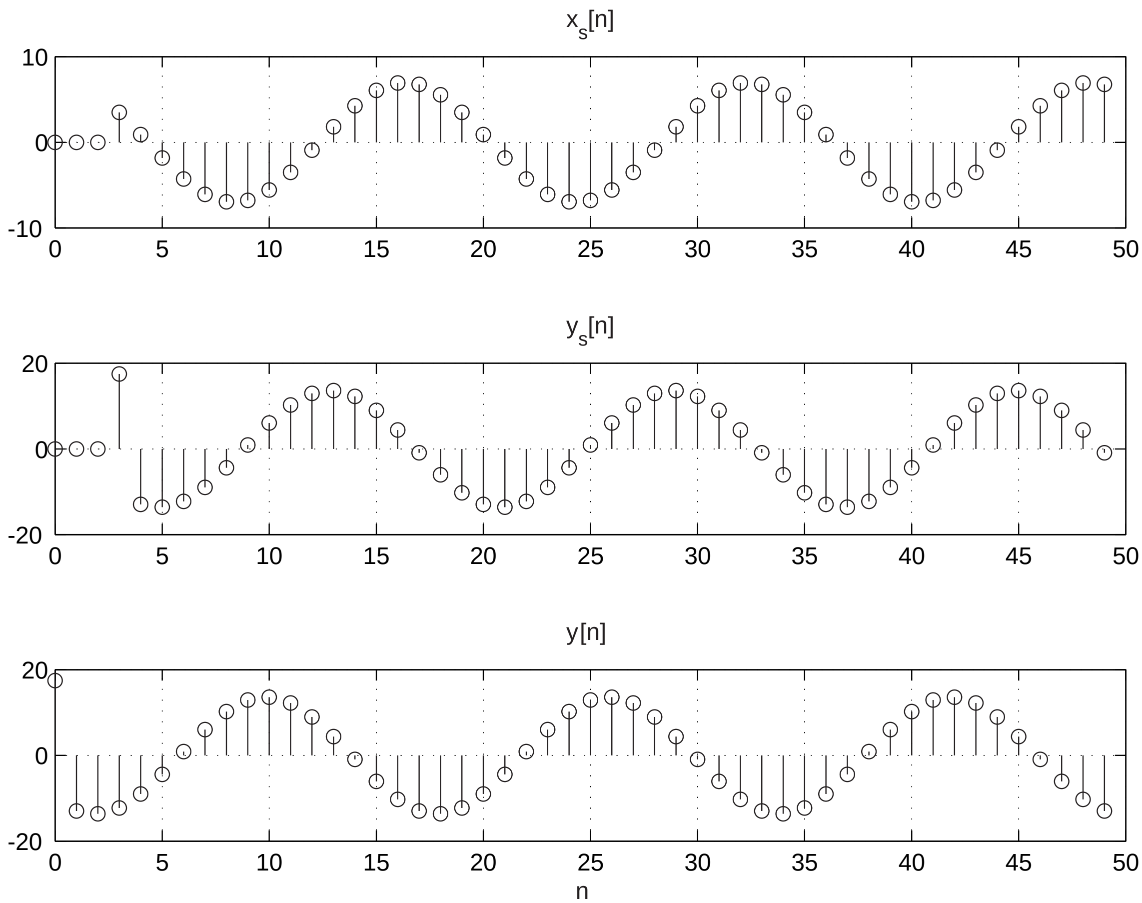

Figure 7: Checking Shift Invariance

(d) **Cascading Two-Systems**

(1)

(2)

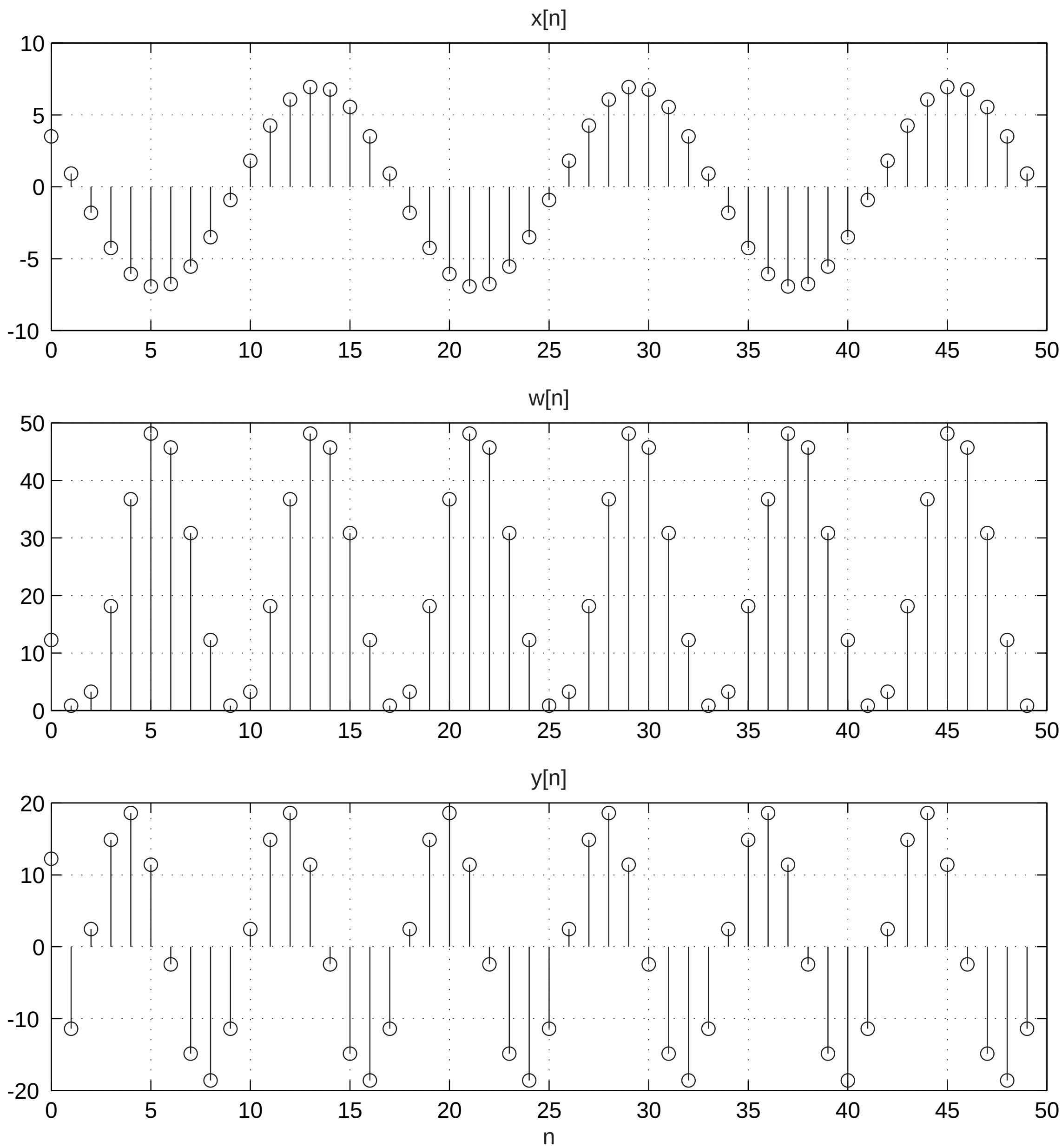

Figure 8: Cascading (Time Domain)

(3) Recall that $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$*. From this equation, we see that the squaring operation doubles the frequency and adds a constant (0 frequency) term as well. (From this equation you can also see that squaring doubles the phase as well.) This frequency doubling is apparent in the plot of* $w[n]$*. Our input* $x[n]$ *has a frequency of* $0.125\pi = \frac{\pi}{8}$*. The squaring process doubles that frequency to* $0.25\pi = \frac{\pi}{4}$*, and adds a constant term (0 frequency), and squares the magnitude as well.*

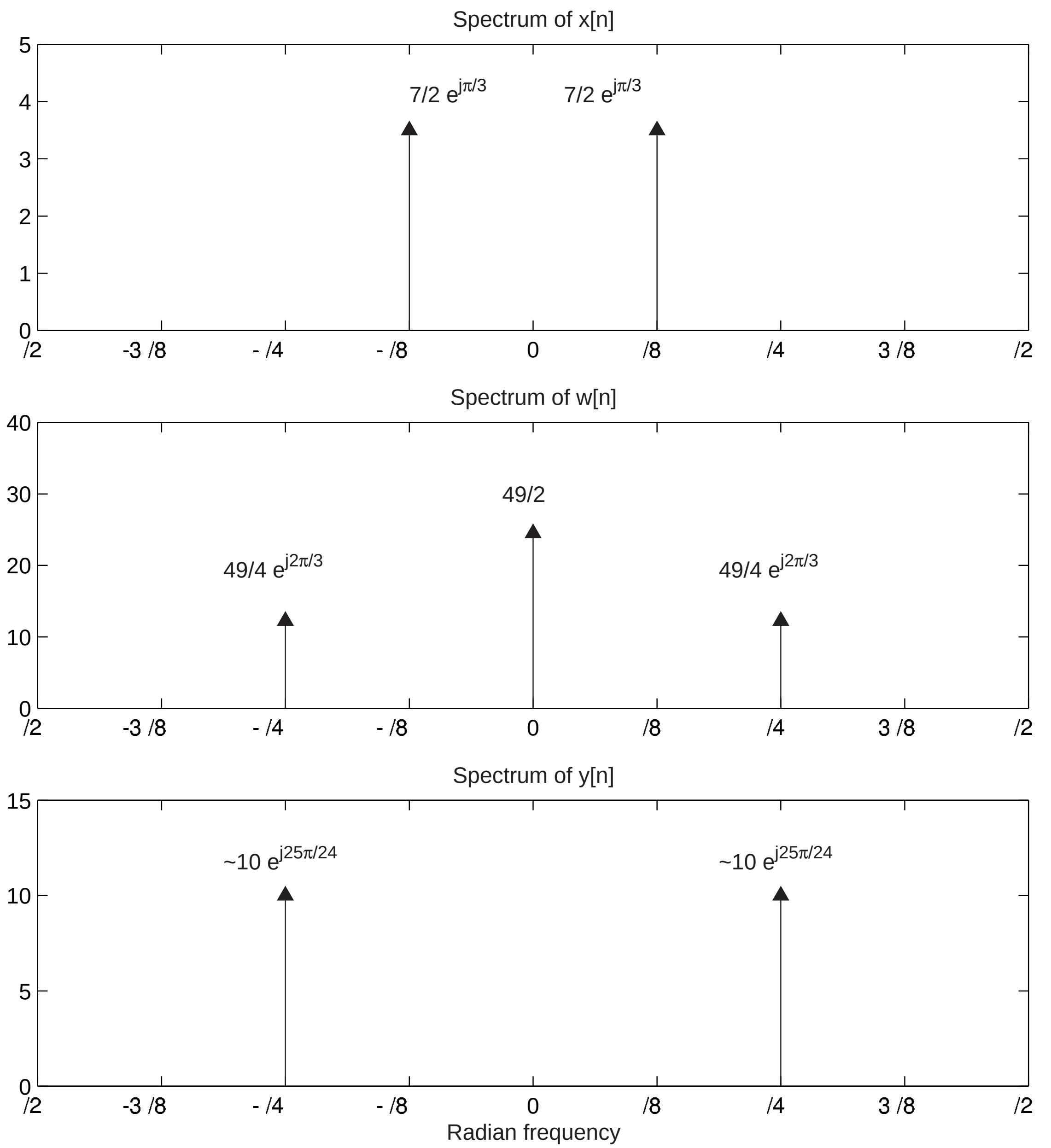

Figure 9: Cascading (Frequency Domain)

(4) Yes, as can be seen in figure 8 the period of $x[n]$ is around 15. The period of $w[n]$ is around 8.

(5) At the frequencies of $w[n]$, the first-difference filter has a magnitude response of:

$$|\mathcal{H}(0)| = 2\sin(0/2) = 0$$
$$|\mathcal{H}(0.25\pi)| = 2\sin(0.25\pi/2) \approx 0.7654$$

So $y[n]$ removes the constant (0 frequency term) and has a magnitude of $(0.7654)(49/2) \approx 20$ due to the frequency of 0.25π in $w[n]$.

As we showed above, the first-difference filter has a linear phase of $-(\hat{\omega} - \pi)/2$. Since our input signal $x[n]$ has a phase of $\frac{\pi}{3}$, the squaring operation doubles it to $\frac{2\pi}{3}$. The first-difference filter then changes the phase to be:

$$\frac{2\pi}{3} - \frac{0.25\pi - \pi}{2} = \frac{25\pi}{24}$$

(6) Now we replace the first-difference filter with the second-order FIR filter $y_2[n] = w[n] - 2cos(0.25\pi)w[n-1] + w[n-2]$. There are no frequencies present in the output signal because the filter is designed to have zero magnitude at $\hat{\omega} = \frac{\pi}{4}$. When the input to the filter is $w[n] = (x[n])^2$, all that remains is the constant (0 frequency) term. When $w[n] = e^{j0.25\pi n}$, the resulting output is zero. The spectrum of $y_2[n]$ in this case is simply zero as well.

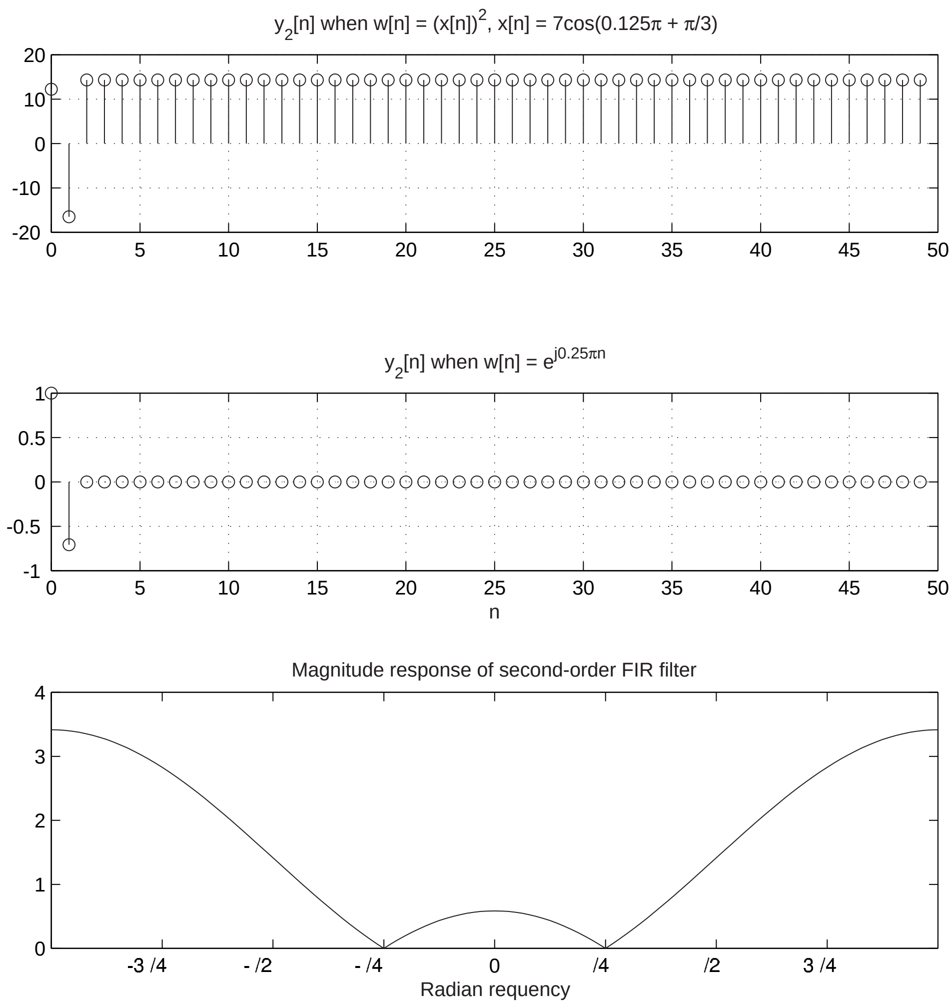

Figure 10: Second Order Differencing

Problem 4: Additional Problem (for MAS.510)

The MATLAB function `zplane` is great for plotting the poles and zeros of a system in the z-plane. Use `zplane` as well as `freqz` to answer the following questions.

(a) Consider the general N-point FIR averaging filter, where each coefficient b_k is simply $\frac{1}{N}$. Plot the zeros of this system in the z-plane as well as the frequency response (magnitude and phase) for $N = 3$, 4, 5, and 10. How does the position of the zeros change? Qualitatively, how does the frequency response change, and how does this relate to the location of the zeros? Does N being an even or odd number have any effect?

(b) Consider the general N-point FIR difference filter, where each coefficient b_k is simply $\frac{(-1)^k}{N}$. Plot the zeros of this system in the z-plane as well as the frequency response (magnitude and phase) for $N = 3$, 4, 5, and 10. How does the position of the zeros change? Qualitatively, how does the frequency response change, and how does this relate to the location of the zeros? Does N being an even or odd number have any effect?

SOLUTION :

(a) *The zeros are evenly spaced around the unit circle (except at $z = 1$, making this a low-pass filter). The more zeros that are added, the narrower the pass region of the filter. If N is even, there is a zero at $z = -1$ (maximum frequency), but if N is odd, there is not.*

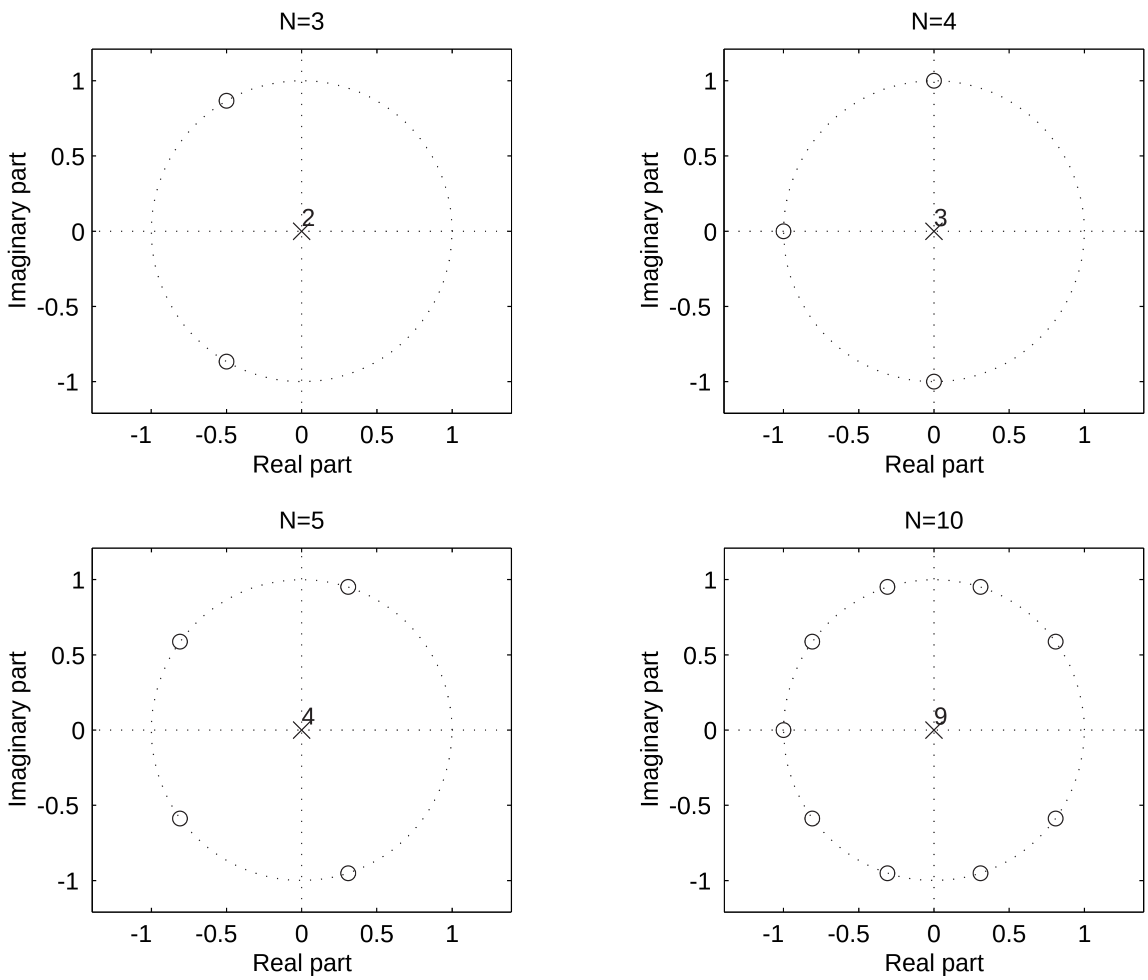

Figure 11: Zeros of Averaging Filters

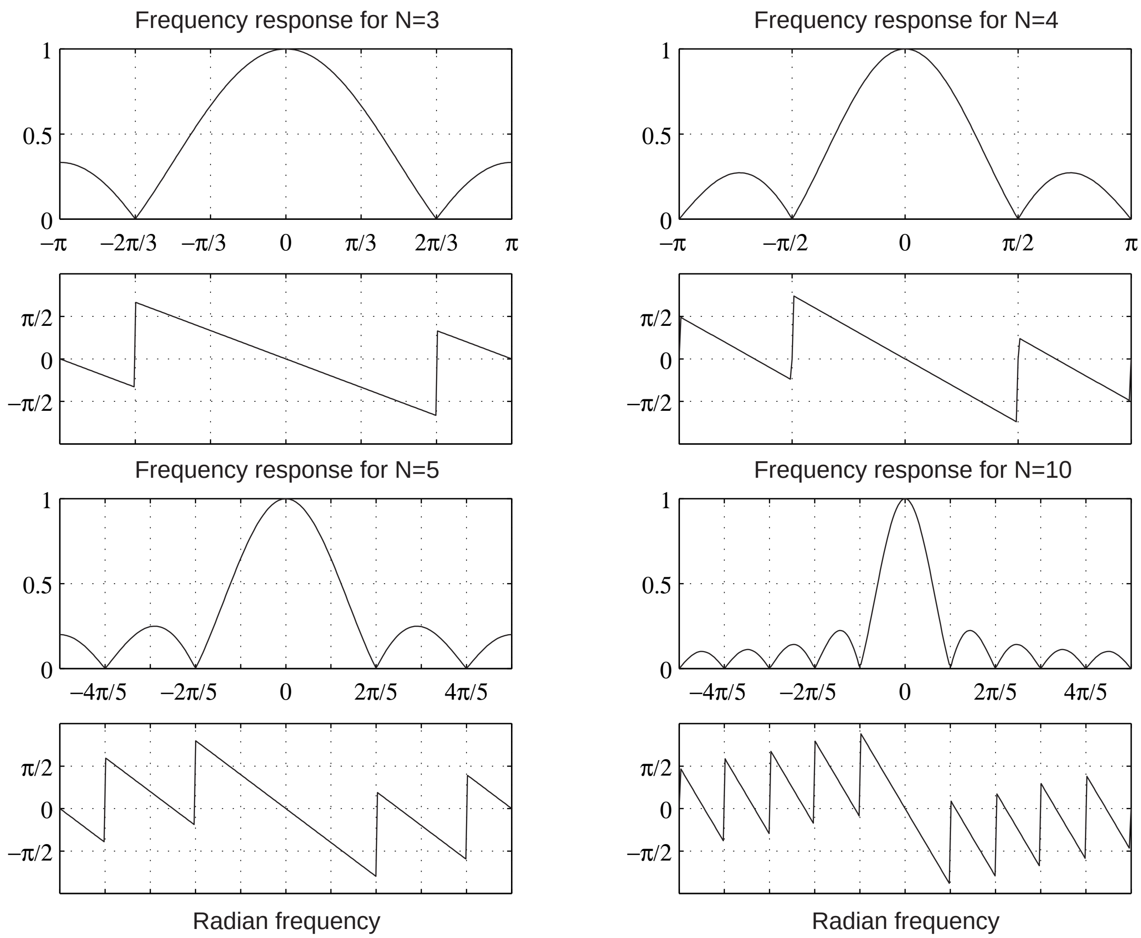

Figure 12: Frequency Response of Averaging Filters

(b) The zeros are evenly spaced around the unit circle (except at $z = -1$, making this a high-pass filter). Again, the more zeros that are added, the narrower the pass region of the filter. If N is even, there is a zero at $z = 1$ (zero frequency), but if N is odd, there is not.

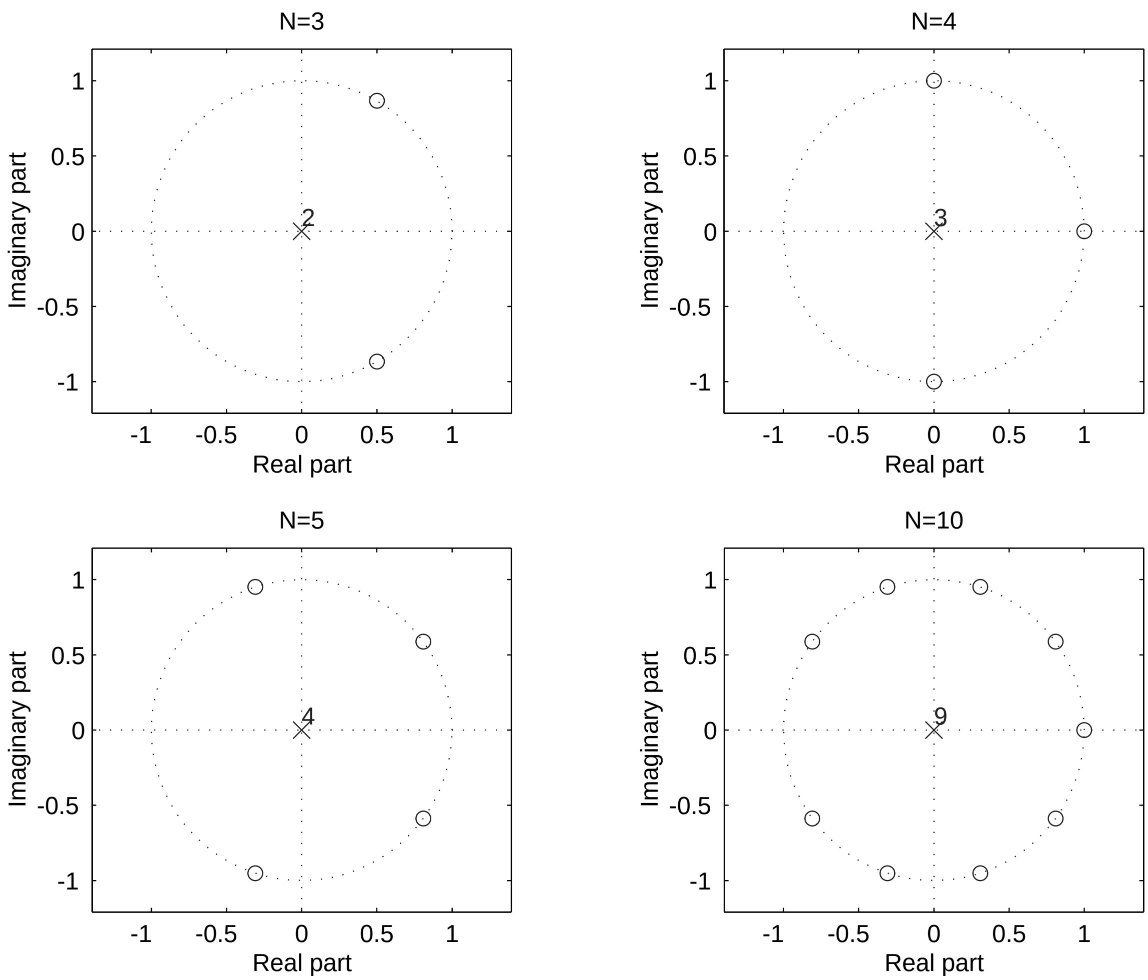

Figure 13: Zeros of Differencing Filters

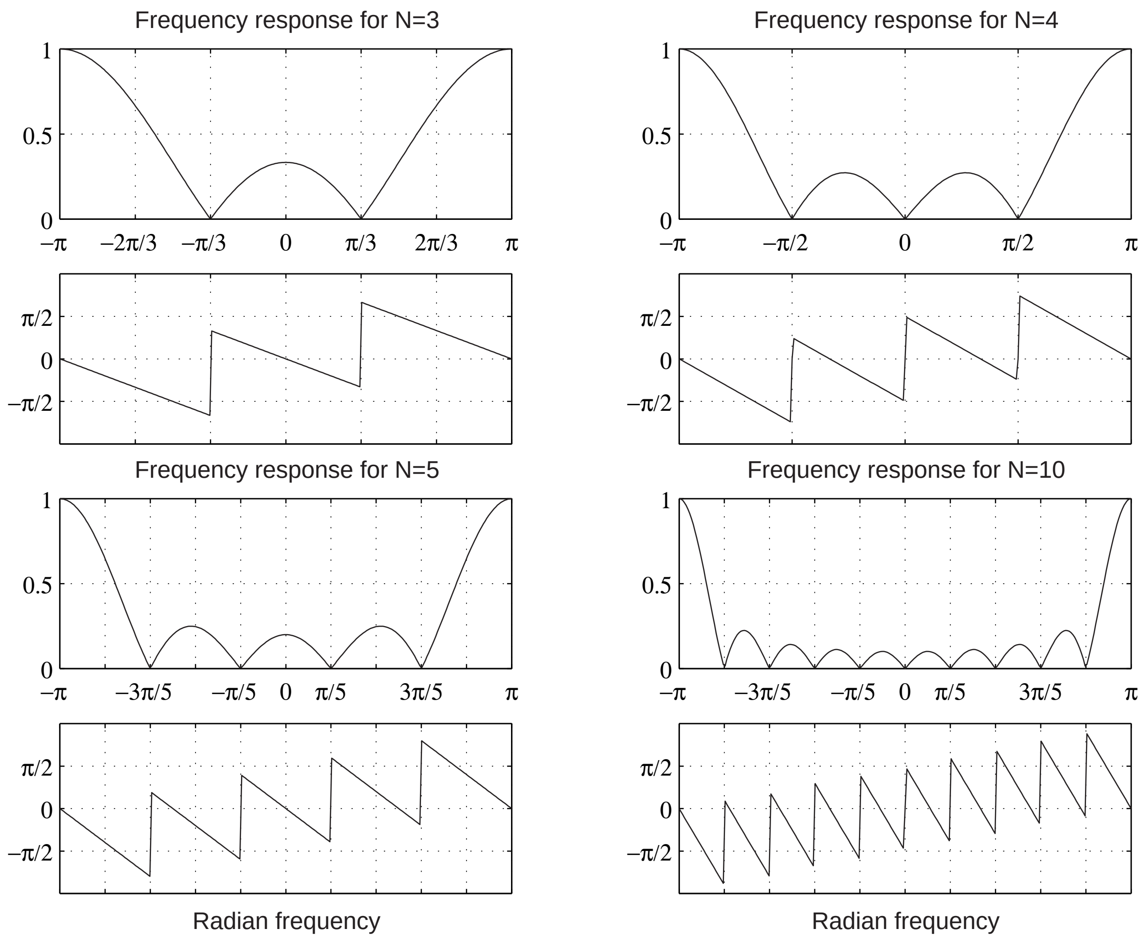

Figure 14: Frequency Response of Averaging Filters