MAS160: Signals, Systems & Information for Media Technology

Problem Set 7

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Problem 1: z-Transforms, Poles, and Zeros

Determine the z-transforms of the following signals. Sketch the corresponding pole-zero patterns.

(a) $x[n] = \delta[n-5]$

(b) $x[n] = nu[n]$

(c) $x[n] = \left(-\frac{1}{3}\right)^n u[n]$

(d) $x[n] = (a^n + a^{-n})u[n], a$ real

(e) $x[n] = (na^n \cos \omega_0 n)u[n], a$ real

(f) $x[n] = \left(\frac{1}{2}\right)^n (u[n-1] - u[n-10])$

SOLUTION :

(a)

$$\begin{aligned}\delta[n-5] &\xrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta[n-5]z^{-n} \\ &= \delta[5-5]z^{-5} \\ &= z^{-5}\end{aligned}$$

ROC : all z*, except* $z = 0$.

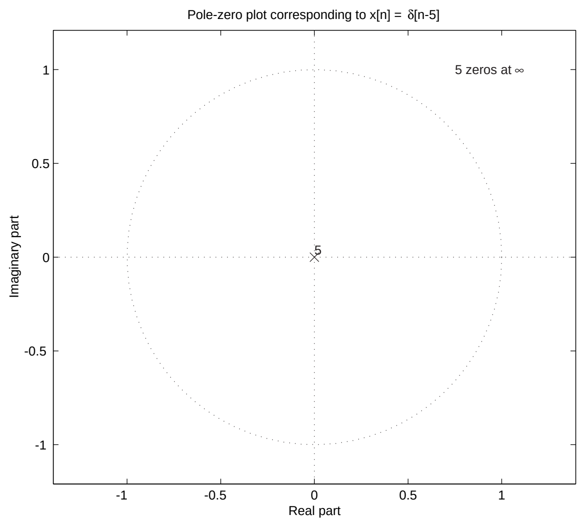

Figure 1: z-plane plot for (a) $x[n] = \delta[n-5]$

(b)

$$nu[n] \xrightarrow{Z} \sum_{n=-\infty}^{\infty} nu[n]z^{-n} = X(z)$$

$$\begin{aligned}
X(z) &= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + ... \\
-z^{-1}X(z) &= -z^{-2} - 2z^{-3} - 3z^{-4} - ... \\
(1-z^{-1})X(z) &= -1 + 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + ... \\
&= -1 + \frac{1}{1-z^{-1}} \\
&= \frac{z^{-1}}{1-z^{-1}} \\
X(z) &= \frac{z^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

$ROC : |z^{-1}| < 1 \Rightarrow |z| > 1$

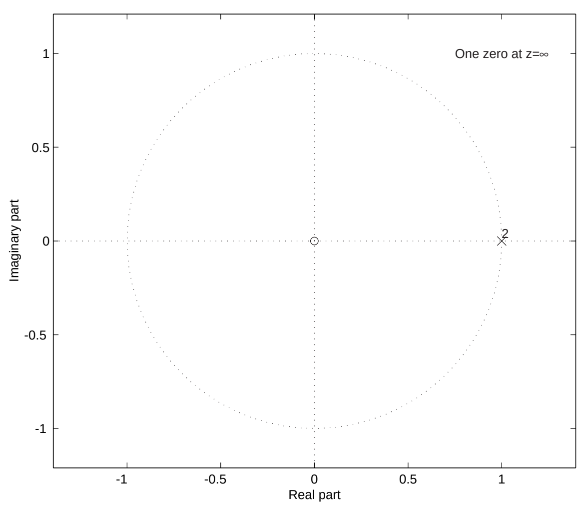

Figure 2: z-plane plot for (b) $x[n] = nu[n]$

(c)

$$\begin{aligned}\left(-\frac{1}{3}\right)^n u[n] \xrightarrow{Z} & \sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}\right)^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n} \\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n \\ &= \frac{1}{1+\frac{1}{3}z^{-1}}\end{aligned}$$

$ROC: |\frac{1}{3}z^{-1}| < 1 \Rightarrow |z| > \frac{1}{3}$

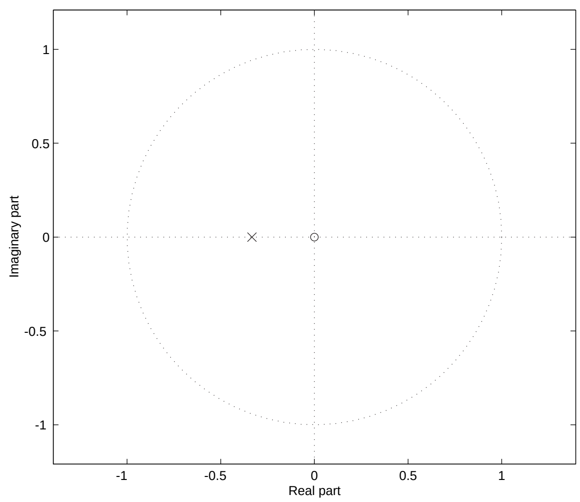

Figure 3: z-plane for (c) $x[n] = \left(-\frac{1}{3}\right)^n u[n]$

(d)

$$
\begin{aligned}
(a^n + a^{-n})u[n] \xrightarrow{Z} & \sum_{n=-\infty}^{\infty} (a^n + a^{-n})u[n]z^{-n} \\
&= \sum_{n=0}^{\infty} (a^n + a^{-n})z^{-n} \\
&= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (a^{-1}z^{-1})^n \\
&= \frac{1}{1 - az^{-1}} + \frac{1}{1 - a^{-1}z^{-1}} \\
&= \frac{1 - a^{-1}z^{-1} + 1 - az^{-1}}{(1 - az^{-1})(1 - a^{-1}z^{-1})} \\
&= \frac{2 - \left(\frac{a^2+1}{a}\right) z^{-1}}{(1 - az^{-1})(1 - a^{-1}z^{-1})}
\end{aligned}
$$

ROC : $|z| > max\ \{a, \frac{1}{a}\}$. *For example, at* $a = 2$*:*

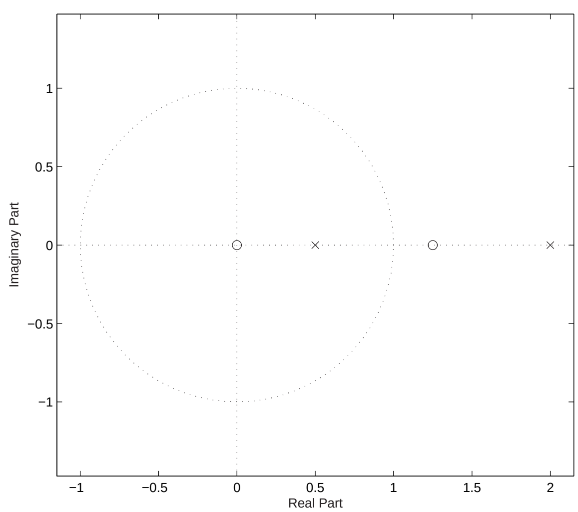

Figure 4: z-plane for (d) $x[n] = (a^n + a^{-n})u[n], a$ real

(e) *Since* $nx[n] \xrightarrow{Z} -z\frac{d}{dz}X(z)$, *first find the* z*-Transform of* $x[n] = a^n \cos(\omega_0 n)u[n]$:

$$
\begin{aligned}
(a^n \cos(\omega_0 n))u[n] \xrightarrow{Z} & \sum_{n=-\infty}^{\infty} (a^n \cos(\omega_0 n))u[n]z^{-n} \\
&= \sum_{n=0}^{\infty} a^n \cos(\omega_0 n) z^{-n} \\
&= \sum_{n=0}^{\infty} a^n \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) z^{-n} \\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty}(ae^{j\omega_0}z^{-1})^n + \sum_{n=0}^{\infty}(ae^{-j\omega_0}z^{-1})^n\right] \\
&= \frac{1}{2}\left[\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{1-ae^{-j\omega_0}z^{-1}}\right] \\
&= \frac{1}{2}\left[\frac{1-ae^{j\omega_0}z^{-1}+1-ae^{-j\omega_0}z^{-1}}{1-ae^{j\omega_0}z^{-1}-ae^{-j\omega_0}z^{-1}+a^2z^{-2}}\right] \\
&= \frac{1}{2}\left[\frac{2-az^{-1}(e^{j\omega_0}+e^{-j\omega_0})}{1-az^{-1}(e^{j\omega_0}+e^{-j\omega_0})+a^2z^{-2}}\right] \\
&= \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}} \\
&= \frac{z^2-az\cos\omega_0}{z^2-2az\cos\omega_0+a^2}
\end{aligned}
$$

Now we take the derivative and multiply by $-z$:

$$
\begin{aligned}
X(z) &= -z\frac{d}{dz}\left(\frac{z^2-az\cos\omega_0}{z^2-2az\cos\omega_0+a^2}\right) \\
&= -z\left(\frac{(z^2-2az\cos\omega_0+a^2)(2z-a\cos\omega_0)-(z^2-az\cos\omega_0)(2z-2a\cos\omega_0)}{(z^2-2az\cos\omega_0+a^2)^2}\right) \\
&= z\left(\frac{z^2a\cos\omega_0-2za^2+a^3\cos\omega_0}{(z^2-2az\cos\omega_0+a^2)^2}\right)
\end{aligned}
$$

We can solve for the poles and zeros using the quadratic formula:

$$\begin{aligned}
\textit{poles: } z &= \frac{2a\cos\omega_0 \pm \sqrt{4a^2\cos^2\omega_0 - 4a^2}}{2} \\
&= a(\cos\omega_0 \pm \sqrt{\cos^2\omega_0 - 1}) \\
&= a(\cos\omega_0 \pm j\sin\omega_0)
\end{aligned}$$

$$\begin{aligned}
\textit{zeros: } z &= 0 \\
z &= \frac{2a^2 \pm \sqrt{4a^4 - 4a^4\cos^2\omega_0}}{2a\cos\omega_0} \\
&= \frac{a \pm a\sin\omega_0}{\cos\omega_0} \\
&= \frac{a(1 \pm \sin\omega_0)}{\cos\omega_0}
\end{aligned}$$

ROC : $|z| > |a|$*. For example, at* $a = \frac{1}{2}$ *and* $\omega_0 = \frac{\pi}{4}$*:*

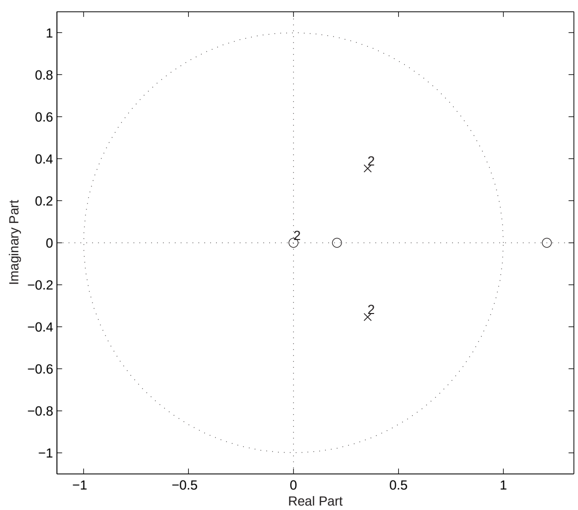

Figure 5: z-plane for (e) $x[n] = (na^n\cos\omega_0 n)u[n], a$ real

(f)

$$\begin{aligned}
\left(\frac{1}{2}\right)^n (u[n-1]-u[n-10]) \xrightarrow{Z} & \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n (u[n-1]-u[n-10]) z^{-n} \\
&= \sum_{n=1}^{9} \left(\frac{1}{2}\right)^n z^{-n} \\
&= \frac{\frac{1}{2}z^{-1} - \left(\frac{1}{2}z^{-1}\right)^{10}}{1-\frac{1}{2}z^{-1}} \\
&= \frac{\frac{1}{2}z^{-1}\left(1-\left(\frac{1}{2}z^{-1}\right)^{9}\right)}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

ROC : The pole and zero at $z = \frac{1}{2}$ *cancel* $\rightarrow$ *all* z*, except* $z = 0$.

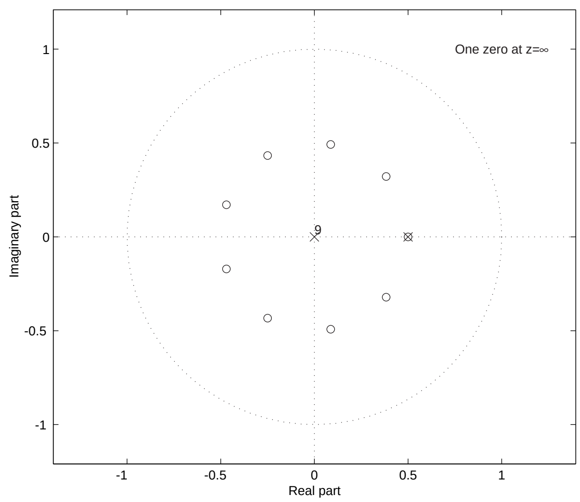

Figure 6: z-plane for (f) $x[n] = \left(\frac{1}{2}\right)^n (u[n-1]-u[n-10])$

Problem 2: z-Transform Properties

Given $x[n]$ below, use the properties of the z-transform to derive the transform of the following signals.

$$x[n] \rightarrow X(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

(a) $x[n-3]$
(b) $x[n] * \delta[n-3]$
(c) $x[n] - x[n-1]$
(d) $x[n] * (\delta[n] - \delta[n-1])$
(e) $5x[n-1] + 4\left(-\frac{1}{3}\right)^n u[n]$

SOLUTION :

(a)

$$\begin{aligned} x[n-3] &\xrightarrow{Z} X(z)z^{-3} \\ &= \frac{z^{-4}}{(1-z^{-1})^2} \end{aligned}$$

(b)

$$\begin{aligned} x[n] * \delta[n-3] &\xrightarrow{Z} X(z)z^{-3} \\ &= \frac{z^{-4}}{(1-z^{-1})^2} \end{aligned}$$

(c)

$$\begin{aligned} x[n] - x[n-1] &\xrightarrow{Z} X(z) - X(z)z^{-1} \\ &= X(z)(1-z^{-1}) \\ &= \frac{z^{-1}}{1-z^{-1}} \end{aligned}$$

(d)

$$\begin{aligned} x[n] * (\delta[n] - \delta[n-1]) &\xrightarrow{Z} X(z)(1-z^{-1}) \\ &= \frac{z^{-1}}{1-z^{-1}} \end{aligned}$$

(e)

$$\begin{aligned} 5x[n-1] + 4\left(-\frac{1}{3}\right)^n u[n] &\xrightarrow{Z} 5X(z)z^{-1} + 4\left(\frac{1}{1+\frac{1}{3}z^{-1}}\right) \qquad \textit{from (1c) above} \\ &= \frac{5z^{-2}}{(1-z^{-1})^2} + \frac{4}{1+\frac{1}{3}z^{-1}} \end{aligned}$$

Problem 3: Relating pole-zero plots to frequency- and impulse-response

(a) *DSP First* 8.16
(b) *DSP First* 8.17

SOLUTION :

(a) *(a) From the magnitude frequency response, we see that A is a high-pass filter, with six zeros along the frequency axis (i.e. the unit circle). Only two of the pole-zero plots have six zeros on the unit circle ($PZ1$ and $PZ2$). From the pole-zero plots, we see $PZ1$ is a low-pass filter (the zeros are concentrated towards higher frequency), while the zeros of $PZ2$ are concentrated towards lower frequencies (making it a high-pass filter). Therefore frequency response A corresponds to pole-zero plot $PZ2$.*

(b) B is a high-pass filter, with a sharp peak near maximum frequency (π) and a zero at zero frequency. This means that there is a pole near the unit circle at $\omega = \pi$ and a zero on the unit circle at $\omega = 0$ ($z = 1$). Therefore, the corresponding pole-zero plot is $PZ5$.

(c) C is a low-pass filter, with six zeros along the frequency axis, corresponding to pole-zero plot $PZ1$.

(d) D is a very sharp band-bass filter, indicating poles close to the unit circle at $\omega = \pm\frac{\pi}{2}$. This is consistent with pole-zero plot $PZ6$.

(e) E is a somewhat complex response, with sharp peaks (indicating poles close to the unit circle) at a low frequency and somewhat smoother peaks at higher frequencies (indicating poles a little bit further from the unit circle). This pattern indicates pole-zero plot $PZ3$.

(b) *(a) The first thing to notice about J is that it is an infinite impulse response, and therefore has a pole somewhere other than at zero or ∞. It's shape (exponential decay), is consistent with a form $h[n] = a^n u[n]$, which is a single-pole system with a pole at $z = a$. And we know from the impulse response that in this case, a is positive, which also indicates a low-pass response. Therefore, the corresponding pole-zero plot is $PZ4$.*

(b) K is FIR, with a length of 7 ($N = 6$) and therefore has six zeros and poles only at zero or ∞. Since each point of the impulse response alternates signs, it is a high-pass filter. All of this leads us to pole-zero plot $PZ2$.

(c) L is IIR, with alternating signs but with zero values in between each alternation. This indicates a band-pass response, centered at a frequency one-half of the maximum frequency (i.e. $\frac{\pi}{2}$). This leads us to pole-zero plot $PZ6$. However, I believe that the correct pole-zero plot would not have a zero at $z = 1$. Using the PeZ tool in MATLAB, if you try plotting the impulse response of a pole-zero pattern corresponding to $PZ6$, you'll get something different, but if you remove the zero at $z = 1$, you obtain impulse response L.

(d) *M is IIR and clearly has a complicated frequency response. From the alternation of signs in the impulse response, we can see that it has both high-pass and band-pass characteristics. Therefore, the corresponding pole-zero plot is $PZ3$.*

(e) *N is FIR ($N = 6$) and is an averaging (low-pass) filter, and thus corresponds to $PZ1$.*

Problem 4: *DSP First* Lab 10

Items to be turned in:

(a) Answers to questions from C.10.4.
(b) Answers to questions from C.10.5.
(c) Plots and answers to questions from C.10.6.

SOLUTION :

(a) *Moving the pole from $z = 0.5$ to the origin changes the impulse response to be just an impulse, creating an all-pass filter (flat magnitude response) that essentially does nothing (like multiplying by 1). Moving the pole closer to the unit circle creates a very sharp low-pass filter and slows the rate of decay of the impulse response. Putting the pole on the unit circle gives us an impulse response of $u[n]$, which is unstable, but gives a very sharp (impulse-like) low-pass filter. Moving the pole outside the unit circle results in an unstable impulse response $h[n]$.*

(b) *If the determinant ($a_1^2 - 4a_2$: the factor under the square root in the quadratic formula) is less than zero, the roots of the polynomial will be a complex conjugate pair.*

$$
\begin{aligned}
H(z) &= \frac{B(z)}{A(z)} \\
&= G\frac{(1 - 0z^{-1})(1 - 0z^{-1})}{(1 - 0.75e^{j\pi/4}z^{-1})(1 - 0.75e^{-j\pi/4}z^{-1})} \\
&= \frac{G}{(1 - 0.75e^{j\pi/4}z^{-1})(1 - 0.75e^{-j\pi/4}z^{-1})}
\end{aligned}
$$

Placing the poles at $z = 0.75e^{\pm j\pi/4}$ results in a band-pass filter, with the pass-band centered at $\pm\pi/4$. Changing the angle of the pole correspondingly changes the location of the pass-band. The variation in the impulse response also increases with increasing pole angle. Increasing the magnitude of the pole makes the pass-band sharper (narrower and taller), and decreases the rate of decay of the impulse response, $h[n]$. Going outside the unit circle, of course, results in an unstable filter.

(c) *A pole at the origin results in an impulse response of a delayed impulse, a flat magnitude response, and a linear phase response. Adding poles at the origin increases the delay in the impulse response (still a delayed impulse), leaves the magnitude response flat, and increases the slope of the phase response.*

Zeros at $z = -1, \pm j$ result in a 4-pt. averaging (low-pass) filter. The phase response is $-\frac{3\omega}{2}$.

To get the the desired FIR response, the zeros should be at $z = \pm j$. To get the desired IIR response, the poles should be at $z = \pm 0.9j$. The cascaded system has the following magnitude response:

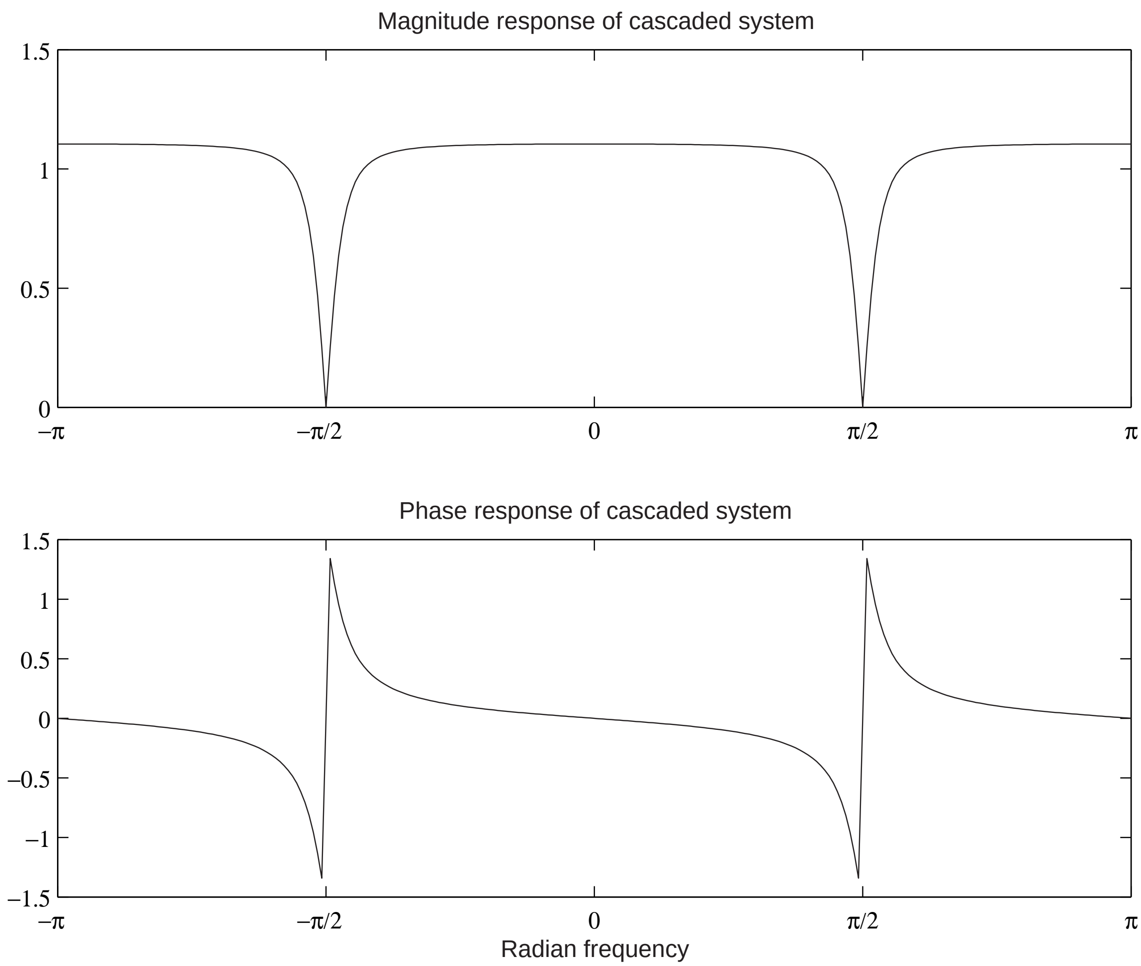

Figure 7: Response of a notch filter

The notches result from the zeros dominating over the poles the closer we get to $\hat{\omega} = \pm\frac{\pi}{2}$. The gain is the same at $\hat{\omega} = 0$ and $\hat{\omega} = \pi$ since those points are equidistant from the zeros and poles.