Problem 1: Return of the rabbits!

Consider the case of the Fibonacci rabbits: They become fertile at one month of age, they have a gestation period of one month, each litter is one male and one female, and they live and breed "forever," from our perspective.

From that description, we could write the following difference equation for the output of a system:

$$r[n] = r[n-1] + r[n-2]$$

where r is pairs of rabbits and n is months, and we will introduce 1 pair of rabbits into the system at $n = 1$(more explicitly, for $n \leq 0$, $r[n] = 0$, and $r[1] = 1$).

(a) Let's think about this a bit more explicitly in z-transform terms. Where are the poles and zeroes of this system?

(b) Suppose we add a new twist: only P percent of the fertile (*i.e.* at-least-two-month-old) rabbits give birth in any given month. Rewrite the difference equation. Now where are the poles and zeroes? *If it makes your life easier, from this point on you can forget that r must always be an integer. If you are looking for a challenge, try to retain that constraint!*

SOLUTION :

(a) *To find the poles and zeroes of the function, we first must find the system function, $R(z)$. We can re-write the difference equation to include the initial conditions (we introduced 1 pair of rabbits into the system at $n = 1$).*

$$r[n] = r[n-1] + r[n-2] + \text{x}[n] \quad \text{x[n]=}\delta\text{[n-1]}$$

From this we can find the system function, $R(z)$:

$$R(z) = R(z)z^{-1} + R(z)z^{-2} + \text{X(z)}$$

$$\text{H(z)=R(z)/X(z)} = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$

From this, we can see that there is are two zeroes at z=0. The poles are a little uglier :

$$z = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

From the way this is set up, we know that the system has to be causal(we cannot have rabbits in the system before we introduce(pun intended) the first pair). Also, we can see that one pole, $\frac{1}{2} + \frac{\sqrt{5}}{2}$, lies outside the unit circle, making the ROC $|z| > \frac{1}{2} + \frac{\sqrt{5}}{2}$, and the system is unstable (which is what we expect with rabbits).

(b) *Introducing the constrain that only P fertile pairs reproduce changes the difference equation to*

$$r[n] = r[n-1] + Pr[n-2] + x[n]$$

which leads to the system function

$$H(z) = \frac{z^2}{z^2 - z - P}$$

and the poles

$$z = \frac{1}{2} \pm \frac{\sqrt{1+4P}}{2}$$

The zeroes do not change(unless $P = 0$ and the system function reduces). Given the constraint that $0 < P < 1$, one pole is still outside the unit circle. The system is still unstable (if we have any number of rabbits reproducing, the system will still blow up). You can never have enough cute bunnies! ;)

Problem 2: Inverse z-Transforms

Determine all possible signals that can have the following z-transforms with the given conditions.

(a) $\frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$

(b) $\frac{2 - \frac{3}{2}z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$, causal

(c) $\frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$, stable

(d) $\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}$, right-handed

SOLUTION :

(a)

$$\frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = \frac{-1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - z^{-1}}$$

At this point, there is no specified ROC, so the inverse z-transform can be any one of three. We have poles at $z = \frac{1}{2}$ and $z = 1$:
For $|z| < \frac{1}{2}$

$$x[n] = -(-1)\left(\frac{1}{2}\right)^n u[-n-1] - (2)u[-n-1]$$

For $\frac{1}{2} < |z| < 1$

$$x[n] = (-1)\left(\frac{1}{2}\right)^n u[n] - (2)u[-n-1]$$

For $|z| > 1$

$$x[n] = (-1)\left(\frac{1}{2}\right)^n u[n] + (2)u[n]$$

(b)

$$\frac{2 - \frac{3}{2}z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - z^{-1}}$$

Since we want a causal system, the only possible(but not guaranteed) ROC would have to be right-sided or $|z| > 1$ so

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + u[n]$$

which does happen to be causal so this is the inverse z-transform.

(c)

$$\frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}} = \frac{-\frac{3}{8}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{27}{8}}{1 - 3z^{-1}}$$

The stable system will be the one which has an ROC that contains the unit circle or $\frac{1}{3} < |z| < 3$ so

$$x[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{27}{8}(3)^n u[-n-1]$$

(d)

$$\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}} = -1 + \frac{2}{1 + \frac{1}{2}z^{-1}}$$

A right-sided system implies that the ROC is $|z| > \frac{1}{2}$ so

$$x[n] = -\delta[n] + 2\left(-\frac{1}{2}\right)^n u[n]$$

Problem 3: Utilizing the z-transform (*DSP First* **8.12**)

SOLUTION :

(a)

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

$$Y(z) = \frac{1}{2}Y(z)z^{-1} + X(z)$$

$$Y(z) - \frac{1}{2}Y(z)z^{-1} = X(z)$$

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

To find the response for the unit-step sequence, $u[n]$, we simply multiply $H(z)$ and $U(z)$, the z-Transform of $u[n]$.

$$Y(z) = H(z)U(z)$$

$$= \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - z^{-1}}$$

$$A = Y(z)\left(1 - \frac{1}{2}z^{-1}\right)\Big|_{z^{-1}=2} = \frac{1}{1 - z^{-1}}\Big|_{z^{-1}=2} = -1$$

$$B = Y(z)\left(1 - z^{-1}\right)\big|_{z^{-1}=1} = \frac{1}{1 - \frac{1}{2}z^{-1}}\Big|_{z^{-1}=1} = 2$$

$$Y(z) = \frac{-1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - z^{-1}}$$

Since $y[n]$ is causal:

$$y[n] = -\left(\frac{1}{2}\right)^n u[n] + 2u[n]$$

(b)

$$X(z) = \frac{1}{1 - e^{j\pi/4}z^{-1}}$$

$$Y(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - e^{j\pi/4}z^{-1}}\right)$$

$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - e^{j\pi/4}z^{-1}}$$

$$A = Y(z)\left(1 - \frac{1}{2}z^{-1}\right)\Big|_{z^{-1}=2} = \frac{1}{1 - e^{j\pi/4}z^{-1}}\Big|_{z^{-1}=2} = \frac{1}{1 - 2e^{j\pi/4}}$$

$$B = Y(z)\left(1 - e^{j\pi/4}z^{-1}\right)\Big|_{z^{-1}=e^{-j\pi/4}} = \frac{1}{1 - \frac{1}{2}z^{-1}}\Big|_{z^{-1}=e^{-j\pi/4}} = \frac{1}{1 - \frac{1}{2}e^{-j\pi/4}}$$

$$Y(z) = \frac{\frac{1}{1 - 2e^{j\pi/4}}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{1 - \frac{1}{2}e^{-j\pi/4}}}{1 - z^{-1}}$$

$$y[n] = \frac{1}{1 - 2e^{j\pi/4}}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{1 - \frac{1}{2}e^{-j\pi/4}}e^{j\pi n/4}u[n]$$

At first, this doesn't look very helpful, but we can see that the first term decays quickly due to the $\left(\frac{1}{2}\right)^n$ term and that the second term is a complex sinusoid due to the $e^{j(\pi/4)n}$ term. Thus, the first term is the transient component and the second term is the steady-state component. If we plot the real and imaginary parts of the signal in MATLAB, we see that the response quickly settles into a steady-state:

(c) From above, we know that the second part of $y[n]$ is the steady state response. We will now compare this to the response of the system at $\hat{\omega} = \frac{\pi}{4}$. To obtain the frequency response of the system, we simply evaluate $H(z)$ at $z = e^{j\hat{\omega}}$:

$$\mathcal{H}(\hat{\omega})|_{\hat{\omega}=\frac{\pi}{4}} = \frac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}}\Big|_{\hat{\omega}=\frac{\pi}{4}}$$

$$= \frac{1}{1 - \frac{1}{2}e^{-j\pi/4}}$$

This is the same as the coefficient to the steady-state response of $y[n]$ and will therefore have the same magnitude and phase. Evaluating these in MATLAB we get:

$$\left|\mathcal{H}\left(\frac{\pi}{4}\right)\right| \approx 1.3572$$

$$\angle\mathcal{H}\left(\frac{\pi}{4}\right) \approx -0.5005$$
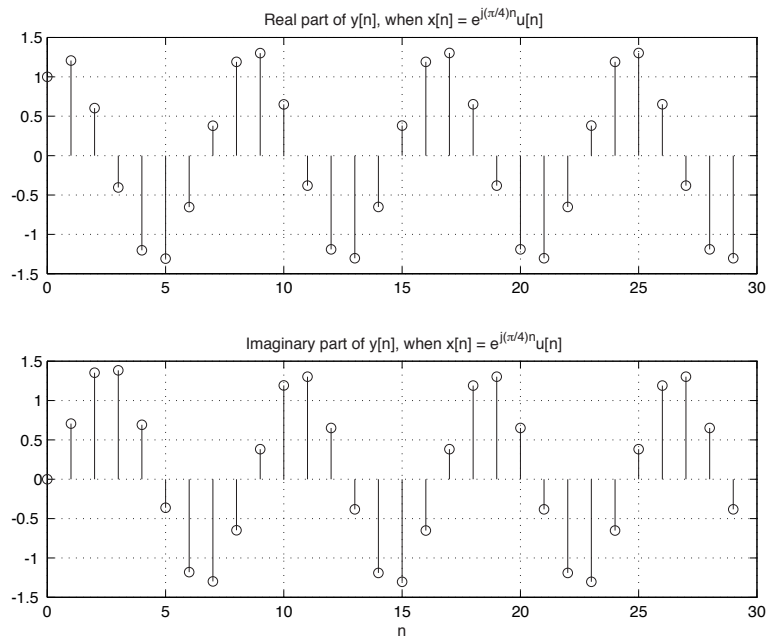
Figure 1: Plots of the Real and Imaginary part of $y[n]$ when $x[n] = e^{j\pi n/4}u[n]$

Problem 4: MAS 510 Additional Problem

(All-Pass System)

Consider the causal linear shift-invariant system with system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}}$$

where a is real

(a) For what range of values of a is the system stable?

(b) If $0 < a < 1$, plot the pole-zero diagram and shade the region of convergence. Do the same for $a > 1$.

(c) $H(z)$ is to be cascaded with a system $\hat{H}(z)$ so that the overall system function is unity. With $0 < a < 1$ and $\hat{H}(z)$ specified to be a stable system, determine its impulse response $\hat{h}(n)$.

SOLUTION :

(a) *Since the system is causal, it is also stable only if the pole lies inside the unit circle. The pole for this system is at $z = a$, so $|a| < 1$*
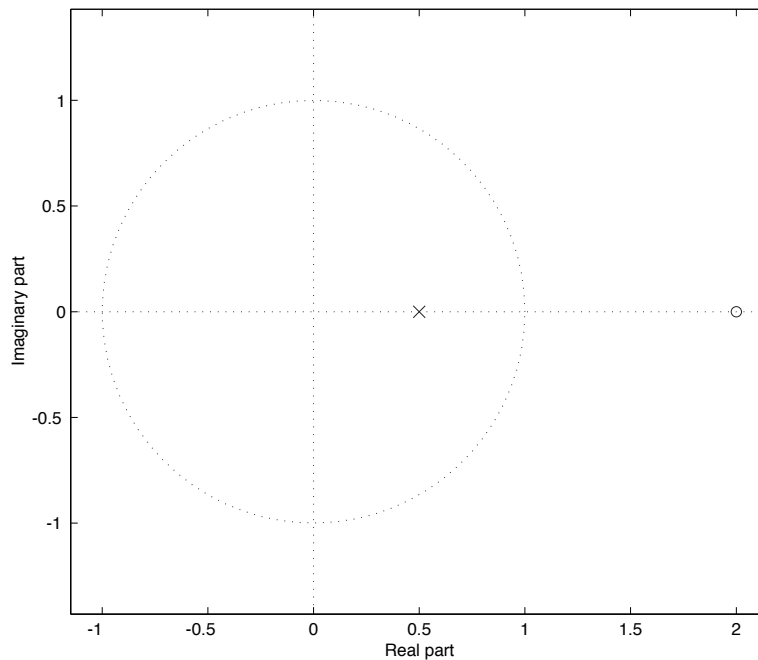
(b) *For $0 < a < 1$*

Figure 2: pole-zero diagram for $0 < a < 1$

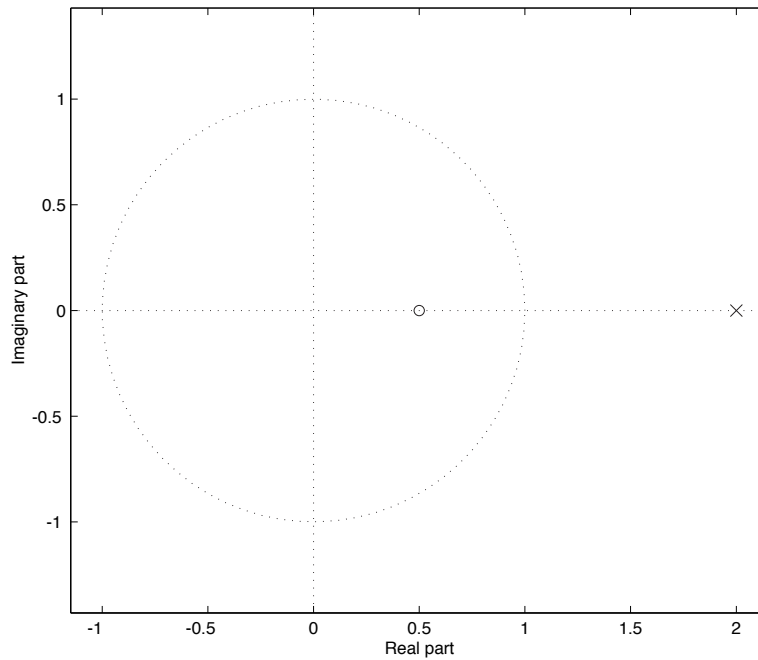*where the shaded region is $|z| > \frac{1}{2}$.*

For $a > 1$

Figure 3: pole-zero diagram for $a > 1$

where the shaded region is $|z| > 2$

(c) *We want $\hat{H}(z)H(z) = 1$ or*

$$\hat{H}(z) = \frac{1}{H(z)} = \frac{1 - az^{-1}}{1 - a^{-1}z^{-1}} = a^2 + \frac{1 - a^2}{1 - a^{-1}z^{-1}}$$

Finding a region of convergence for a stable system with $0 < a < 1$ yields $|z| < |a^{-1}|$. This leads to

$$\hat{h}[n] = a^2\delta[n] - (1 - a^2)(a^{-1})^n u[-n - 1]$$

Problem 5: Discrete Fourier Transforms (*DSP First* **9.2**)

SOLUTION :

$$X_0[k] = \sum_{n=0}^{9} x_0[n]e^{-j(2\pi/10)kn} \tag{a}$$
$$= (1)e^{-j(2\pi/10)k0}$$
$$= 1 \qquad for \ k = 0, 1, 2, ..., 9$$

$$X_1[k] = \sum_{n=0}^{9} x_1[n]e^{-j(2\pi/10)kn} \tag{b}$$
$$= \sum_{n=0}^{9} (1)e^{-j(2\pi/10)kn}$$
$$= \frac{e^{-j(2\pi/10)k0} - e^{-j(2\pi/10)k10}}{1 - e^{-j(2\pi/10)k}}$$
$$= \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi/10)k}}$$
$$= 0 \qquad k \neq 0$$
$$X_1[0] = \sum_{n=0}^{9} (1)e^{-j(2\pi/10)0n}$$
$$= \sum_{n=0}^{9} (1)(1)$$
$$= 10$$
$$X_1[k] = \begin{cases} 10, & k = 0 \\ 0, & k = 1, 2, ..., 9 \end{cases}$$

$$X_2[k] = \sum_{n=0}^{9} x_2[n]e^{-j(2\pi/10)kn} \tag{c}$$
$$= e^{-j(2\pi/10)k4}$$
$$= e^{-j(2\pi/5)2k}$$

$$X_3[k] = \sum_{n=0}^{9} x_3[n] e^{-j(2\pi/10)kn} \tag{d}$$

$$= \sum_{n=0}^{9} e^{-j2\pi n/5} e^{-j(2\pi/10)kn}$$

$$= \sum_{n=0}^{9} e^{-j(2\pi/10)(k-2)n}$$

$$= \frac{e^{-j(2\pi/10)(k-2)0} - e^{-j(2\pi/10)(k-2)10}}{1 - e^{-j(2\pi/10)(k-2)}}$$

$$= \frac{1 - e^{-j2\pi(k-2)}}{1 - e^{-j(2\pi/10)(k-2)}}$$

$$= 0 \qquad k \neq 2$$

$$X_3[2] = \sum_{n=0}^{9} (1) e^{-j(2\pi/10)(2-2)n}$$

$$= \sum_{n=0}^{9} (1)(1)$$

$$= 10$$

$$X_3[k] = \begin{cases} 10, & k = 2 \\ 0, & k \neq 2 \end{cases}$$

Problem 6: Inverse DFT (*DSP First* **9.3**)

SOLUTION :

$$x_a[n] = \frac{1}{10} \sum_{k=0}^{9} X_a[n] e^{j(2\pi/10)kn} \tag{a}$$

$$= \frac{1}{10}(1)e^{-j(2\pi/10)k0}$$

$$= \frac{1}{10} \qquad for \ k = 0, 1, 2, ..., 9$$

$$x_b[n] = \frac{1}{10} \sum_{k=0}^{9} X_b[n] e^{j(2\pi/10)kn} \tag{b}$$

$$= \frac{1}{10} \sum_{k=0}^{9} (1)e^{j(2\pi/10)kn}$$

$$= \frac{1}{10} \frac{e^{j(2\pi/10)0n} - e^{j(2\pi/10)10n}}{1 - e^{j(2\pi/10)n}}$$

$$= \frac{1 - e^{j2\pi n}}{1 - e^{j(2\pi/10)n}}$$

$$= 0 \qquad n \neq 0$$

$$x_b[0] = \frac{1}{10} \sum_{k=0}^{9} (1)e^{j(2\pi/10)k0}$$

$$= \frac{1}{10} \sum_{k=0}^{9} (1)(1)$$

$$= 1$$

$$x_b[k] = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, ..., 9 \end{cases}$$

$$x_c[n] = \frac{1}{10} \sum_{k=0}^{9} X_c[n] e^{j(2\pi/10)kn} \tag{c}$$

$$= \frac{1}{10}(e^{j(2\pi/10)3n} + e^{j(2\pi/10)7n})$$

$$= \frac{e^{j\pi n}}{10}(e^{-j2\pi n/5} + e^{j2\pi n/5})$$

$$= \frac{e^{j\pi n}}{5} \cos(2\pi n/5)$$

$$= \frac{(-1)^n}{5} \cos(2\pi n/5)$$

$$x_d[n] = \frac{1}{10} \sum_{k=0}^{9} X_d[n] e^{j(2\pi/10)kn} \tag{d}$$

$$= \frac{1}{10} \sum_{k=0}^{9} \cos(2\pi k/5) e^{j(2\pi/10)kn}$$

$$= \frac{1}{10} \sum_{k=0}^{9} \frac{1}{2} \left(e^{j2\pi k/5} + e^{-j2\pi k/5} \right) e^{j(2\pi/10)kn}$$

$$= \frac{1}{20} \left(\sum_{k=0}^{9} e^{j2\pi k/5} e^{j(2\pi/10)kn} + \sum_{k=0}^{9} e^{-j2\pi k/5} e^{j(2\pi/10)kn} \right)$$

$$= \frac{1}{20} \left(\sum_{k=0}^{9} e^{j(2\pi/10)k(n+2)} + \sum_{k=0}^{9} e^{j(2\pi/10)k(n-2)} \right)$$

$$= \frac{1}{20} \left(\frac{e^{j(2\pi/10)(n+2)0} - e^{j(2\pi/10)(n+2)10}}{1 - e^{j(2\pi/10)(n+2)}} + \frac{e^{j(2\pi/10)(n-2)0} - e^{j(2\pi/10)(n-2)10}}{1 - e^{j(2\pi/10)(n-2)}} \right)$$

$$= \frac{1}{20} \left(\frac{1 - e^{j2\pi(n+2)}}{1 - e^{j(2\pi/10)(n+2)}} + \frac{1 - e^{j2\pi(n-2)}}{1 - e^{j(2\pi/10)(n-2)}} \right)$$

$$= 0 \qquad n \neq 2, 8$$

$$x_d[2] = \frac{1}{20} \left(\sum_{k=0}^{9} e^{j2\pi k/5} e^{j(2\pi/10)k2} + \sum_{k=0}^{9} e^{-j2\pi k/5} e^{j(2\pi/10)k2} \right)$$

$$= \frac{1}{20} \left(\sum_{k=0}^{9} e^{j2\pi 2k/5} + \sum_{k=0}^{9} e^{-j2\pi k/5} e^{j(2\pi k/5)} \right)$$

$$= \frac{1}{20} \left(\frac{1 - e^{j2\pi 4}}{1 - e^{-j2\pi 2/5}} + \sum_{k=0}^{9} (1) \right)$$

$$= \frac{1}{20} (0 + 10)$$

$$= \frac{1}{2}$$

$$x_d[8] = \frac{1}{20} \left(\sum_{k=0}^{9} e^{j2\pi k/5} e^{j(2\pi/10)k8} + \sum_{k=0}^{9} e^{-j2\pi k/5} e^{j(2\pi/10)k8} \right)$$

$$= \frac{1}{20} \left(\sum_{k=0}^{9} e^{j2\pi k/5} e^{-j2\pi k/5} + \sum_{k=0}^{9} e^{j2\pi 3k/5} \right)$$

$$= \frac{1}{20} \left(\sum_{k=0}^{9} (1) + \frac{1 - e^{j2\pi 6}}{1 - e^{-j2\pi 3/5}} \right)$$

$$= \frac{1}{20} (10 + 0)$$

$$= \frac{1}{2}$$

$$x_d[n] = \begin{cases} \frac{1}{2}, & n = 2, 8 \\ 0, & n \neq 2, 8 \end{cases} \qquad \text{PS 8-12}$$

Problem 7: Convolution revisited

(a) In MATLAB, create two vectors as follows:

```
x1 = [1 1 1 1 0 0 0 0];
x2 = [0 0 1 1 1 1 0 0];
```

Now convolve the two using the `conv` function and plot the result using `stem`.

(b) Now create two new vectors as follows:

```
x3 = [1 1 1 1 1 0 0 0];
x2 = [0 0 1 1 1 1 1 0];
```

Again convolve the two using the `conv` function and plot the result using `stem`.

(c) Remember convolution in the time domain is equivalent to multiplication in the Fourier (frequency) domain. So now calculate the convolution from part (a) by taking the FFT of both vectors (using the `fft` function), multiplying the FFTs (using `.*`), and inverse transforming back to the time domain (using the `ifft` function). You may need to take the `real` part of the inverse transform since very small numerical errors will accumulate in MATLAB during the FFT and IFFT process. How does your result compare with part (a)?

(d) Repeat this procedure for the vectors in part (b). How does your result compare with part (b)? Explain what appears to be happening.
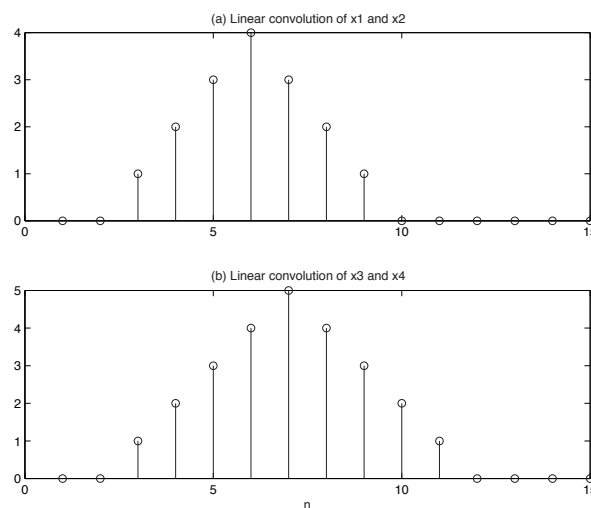*Hint 1: We've seen a similar phenomenon earlier in the semester.*
Hint 2: What is the FFT/DFT doing in the frequency domain?

SOLUTION :
(a)
(b)

(c) *In part(a) the convolution performed using the* `conv` *function is a linear convolution. But the resulting convolution from multiplying the FFTs of* `x1` *and* `x2` *and taking the inverse FFT is a circular convolution. In this case, the initial vectors are long enough so that all of the points from the convolution are preserved. But we can see that one of the points from the linear convolution has wrapped around to the beginning of the circular convolution.*
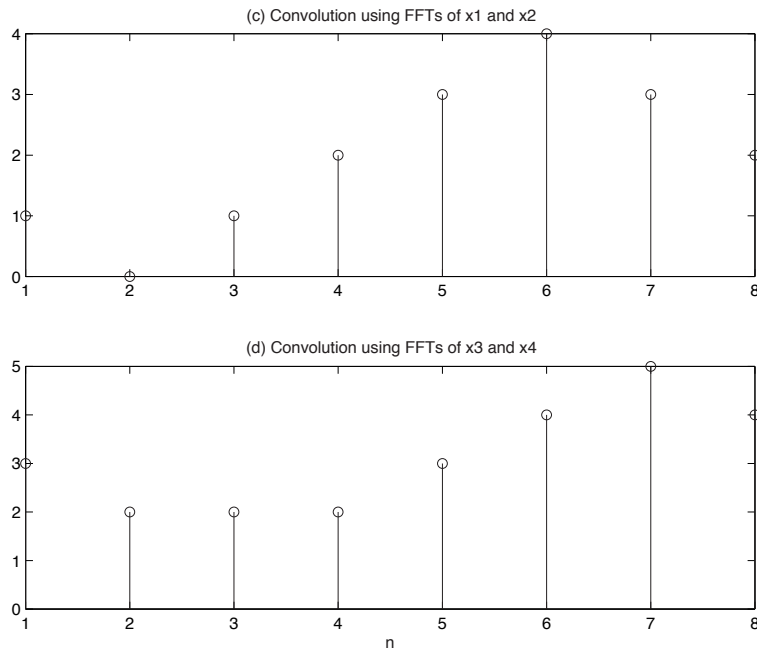

Figure 4: Convolution using FFT

(d) *This time, the initial vectors weren't long enough and the circular convolution has caused some points to overlap. In this case, there is no easy way to recover the linear convolution from the circular convolution. This is called time-domain aliasing. Like aliasing in the frequency domain (which we studied earlier in the semester), some information has been lost. This time, we don't have enough samples (DFT/FFT points) in the frequency domain. A linear convolution of two vectors of length N can be length $2N - 1$, but in this case the FFT and the resulting circular convolution is only of length N. The easiest way to avoid time-domain aliasing when performing convolutions is to zero-pad your input vectors to be long enough (i.e. at least length $2N - 1$) so that the resulting convolution will not alias in the time-domain.*

Problem 8: MAS 510 Additional Problem

Start in MATLAB with the following commands:

```
n = 0:31;
x = cos(2*pi*0.11*n);
X = abs(fft(x));
```

(a) Explain what the last line accomplishes. Plot x and X using `stem`. What information does the plot of X convey? Does it look like what you'd expect?

(b) Now zero-pad x to a length of 2048 using the following command:

```
x1 = [x zeros(1,2016)];
```

Again take the FFT and plot the magnitude of the result. Explain what is going on.

(c) Now window x with a Hanning window and zero-pad to a length of 2048 using the following command:

```
x2 = [x.*hanning(32)' zeros(1,2016)];
```

Take the FFT of x2 and plot the magnitude of the result. Compare this to your plot from part(b). How is it similar and how is it different? Explain what is going on.

SOLUTION :

(a) *The line* `X = abs(fft(x));` *returns the magnitude of the FFT of the vector* `x`.

Since `x` *is a simple cosine, we'd like to see two spikes (at the positive and negative frequencies of the cosine) in the frequency domain. We sort of see that, but we have a lot of other points which don't correspond to the cosine. This is because* `x` *is finite in duration, so we are looking at the frequency representation of a windowed signal. On top of that, the FFT is* samples *of the DTFT, and the samples we've taken look a bit odd.*

(b) *When we zero-pad, we increase the number of samples of the FFT, so we have a much closer approximation to the DTFT. We see that the finite-duration cosine (multiplication of a cosine by a rectangular window) results in the convolution of the DTFT of the cosine (two spikes) and the DTFT of the rectangular window (a sinc).*

(c) *When we window (multiply)* `x` *with a Hamming window, we again convolve in the frequency domain. However, we see that the DTFT of the Hamming window has somewhat different characteristics from a rectangular window. The side lobes (bumps off to the side) are much smaller in magnitude, but the main lobe (the center bump) is wider than that of the rectangular window. That is the tradeoff of using the Hamming window: less stuff off to the side, but less resolution in the middle.*
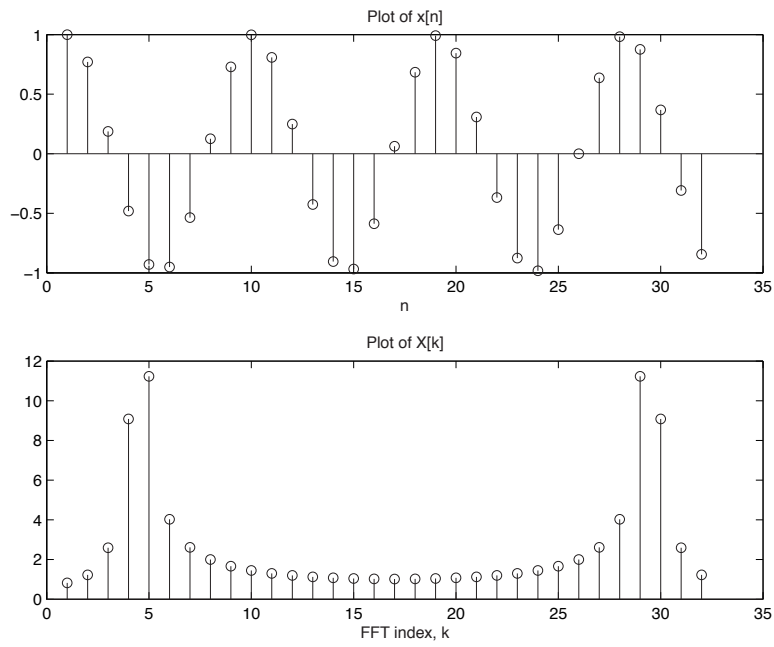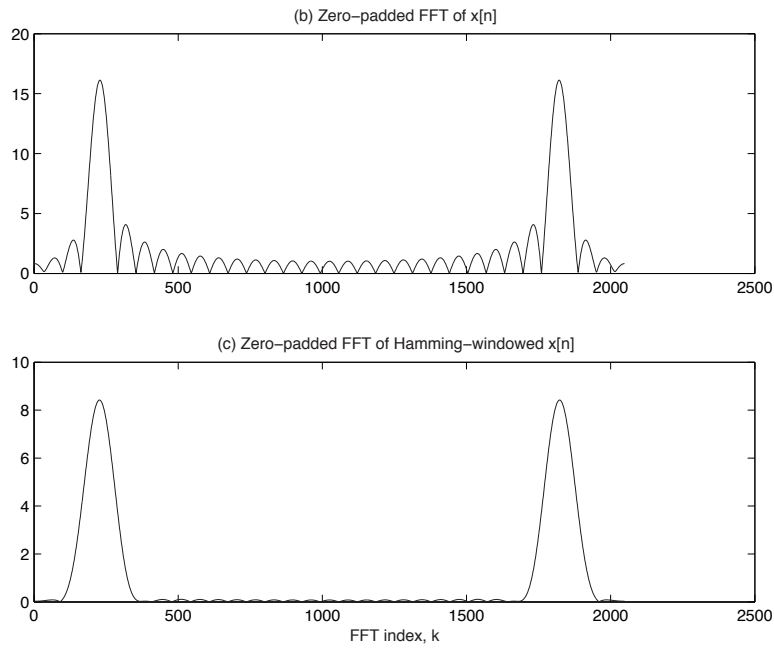
Figure 5: plot of x[n] and X[k] = `abs(fft(x))`

Figure 6: plot of x[n] and X[k] = `abs(fft(x))` with zero-padding