

MAS160: Signals, Systems & Information for Media Technology

Problem Set 6

Instructor : V. Michael Bove, Jr.

Problem 1: Frequency response of FIR filters (DSP First 6.4)

SOLUTION :

(a)

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^M h[k]e^{-j\hat{\omega}k} \\ &= 2e^{-j\hat{\omega}0} - 3e^{-j\hat{\omega}1} + 2e^{-j\hat{\omega}2} \\ &= 2 - 3e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}} (2e^{-j\hat{\omega}} - 3 + 2e^{-j2\hat{\omega}}) \\ &= e^{-j\hat{\omega}} (-3 + 4 \cos(\hat{\omega}))\end{aligned}$$

$$\begin{aligned}|\mathcal{H}(\hat{\omega})| &= |-3 + 4 \cos(\hat{\omega})| \\ \angle \mathcal{H}(\hat{\omega}) &= -\hat{\omega}\end{aligned}$$

(b) $\mathcal{H}(\hat{\omega})$ has a period of 2π .

(c) See figure 1

(d)

$$\begin{aligned}-3 + 4 \cos(\hat{\omega}) &= 0 \\ \cos(\hat{\omega}) &= \frac{3}{4} \\ \hat{\omega} &= \cos^{-1}\left(\frac{3}{4}\right) \\ &\approx \pm 0.7227 \pm 2\pi k \text{ radians, } k \text{ integer}\end{aligned}$$

(e) Since $x[n] = \sin(\frac{\pi}{13}n)$, we need only evaluate the frequency response $\mathcal{H}(\hat{\omega})$ at $\hat{\omega} = \frac{\pi}{13}$:

$$\begin{aligned}\mathcal{H}\left(\frac{\pi}{13}\right) &= e^{-j\pi/13} \left(-3 + 4 \cos\left(\frac{\pi}{13}\right)\right) \\ &\approx 0.8838e^{-j\pi/13}\end{aligned}$$

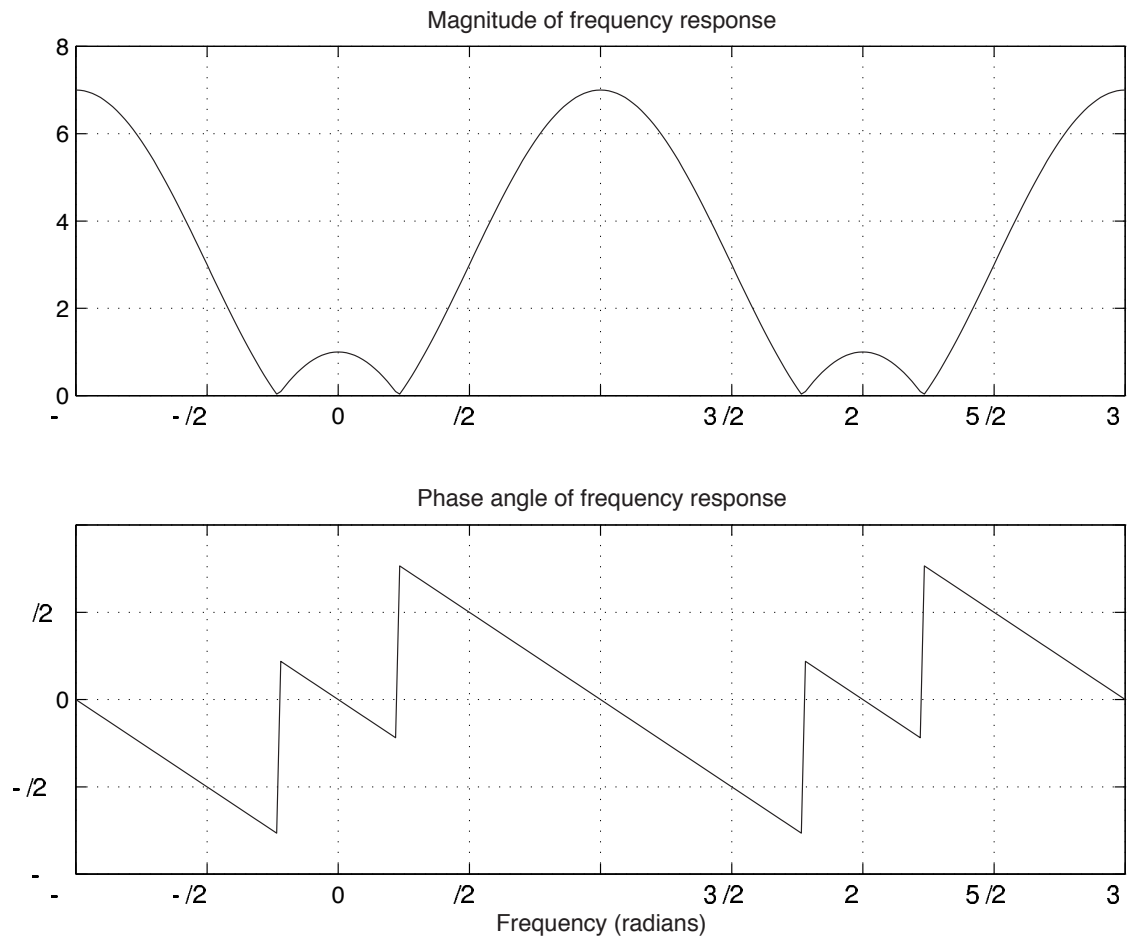


Figure 1: Frequency Response of $y[n] = 2x[n] - 3x[n - 1] + 2x[n - 2]$

Therefore, the magnitude is $|\mathcal{H}(\frac{\pi}{13})| \approx 0.8838$ and the phase is $\angle|\mathcal{H}(\frac{\pi}{13})| = -\frac{\pi}{13}$, so the output $y[n]$ is:

$$\begin{aligned}
 y[n] &= 0.8838 \sin\left(\frac{\pi}{13}n - \frac{\pi}{13}\right) \\
 &= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{\pi}{13} - \frac{\pi}{2}\right) \\
 &= 0.8838 \cos\left(\frac{\pi}{13}(n - 1) - \frac{\pi}{2}\right) \\
 &= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{15\pi}{26}\right)
 \end{aligned}$$

Problem 2: Simple sound filtering

Using our old friend the `sumcos` function, create a sound with a fundamental frequency of 440 Hz, with 12 harmonics of equal amplitude and zero phase, and using the following parameters:

```
fs = 11025;           % Sets sampling rate to 11025 Hz
f = 440*[1:12];      % Creates frequency vector of 12 harmonics of 440 Hz
X = ones(1,12);      % Creates amplitudes of 1
dur = 1;             % Sets duration to be 1 sec
```

Use MATLAB to perform the following tasks:

- (a) Create a three-point averaging FIR filter and plot the frequency response (magnitude and phase) of this filter using `freqz`. What is this filter supposed to do? Filter the sound you created above with this filter. How does it compare to the original sound?
- (b) Create a two-point first difference FIR filter and plot the frequency response (magnitude and phase) of this filter using `freqz`. What is this filter supposed to do? Filter the original sound you created above using this new filter. How does it compare to the original sound?

SOLUTION :

(a) *This is a low-pass filter designed to filter out high frequencies. You should hear it attenuate the higher frequencies in your sound, much like turning down the treble control on your stereo.*

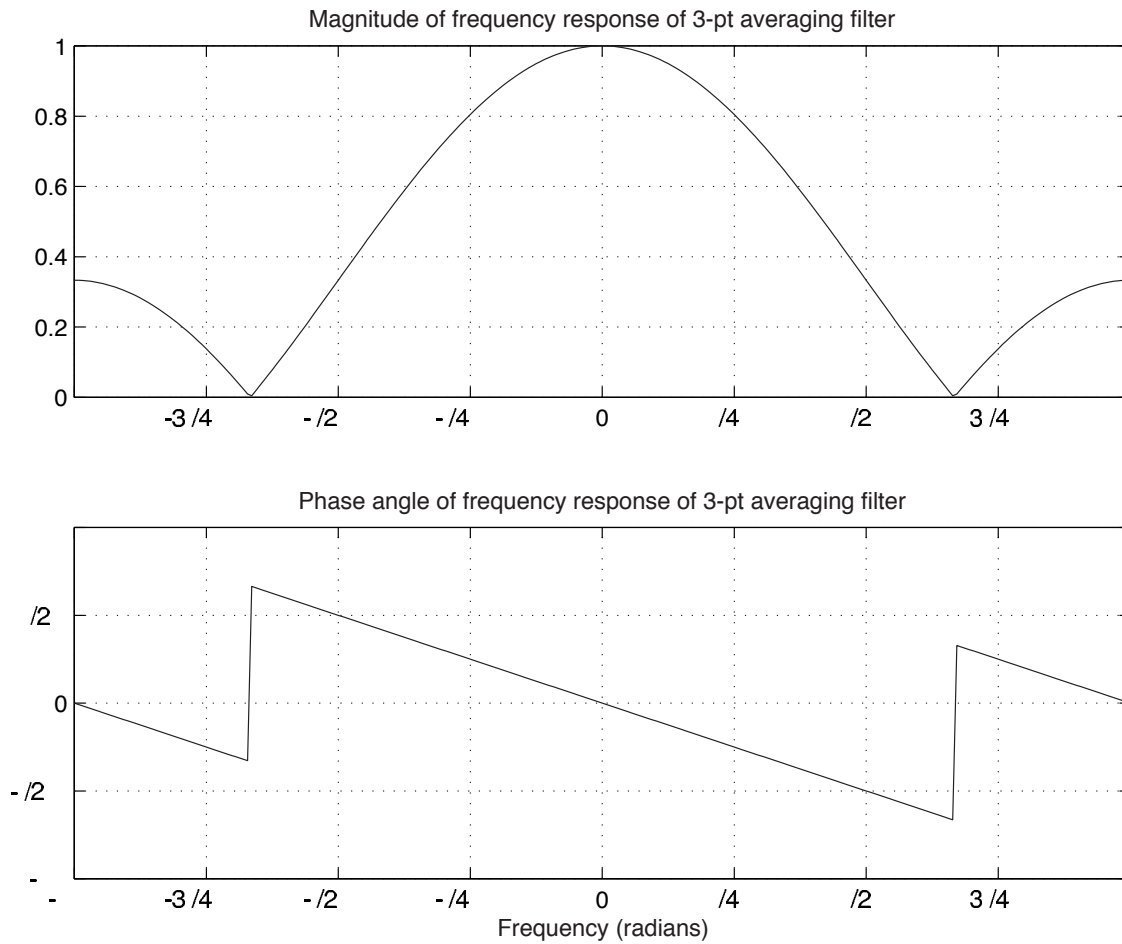


Figure 2: Averaging Filter

(b) This is a high-pass filter designed to filter out low frequencies. You should hear it attenuate the lower frequencies in your sound, much like turning down the bass control on your stereo.

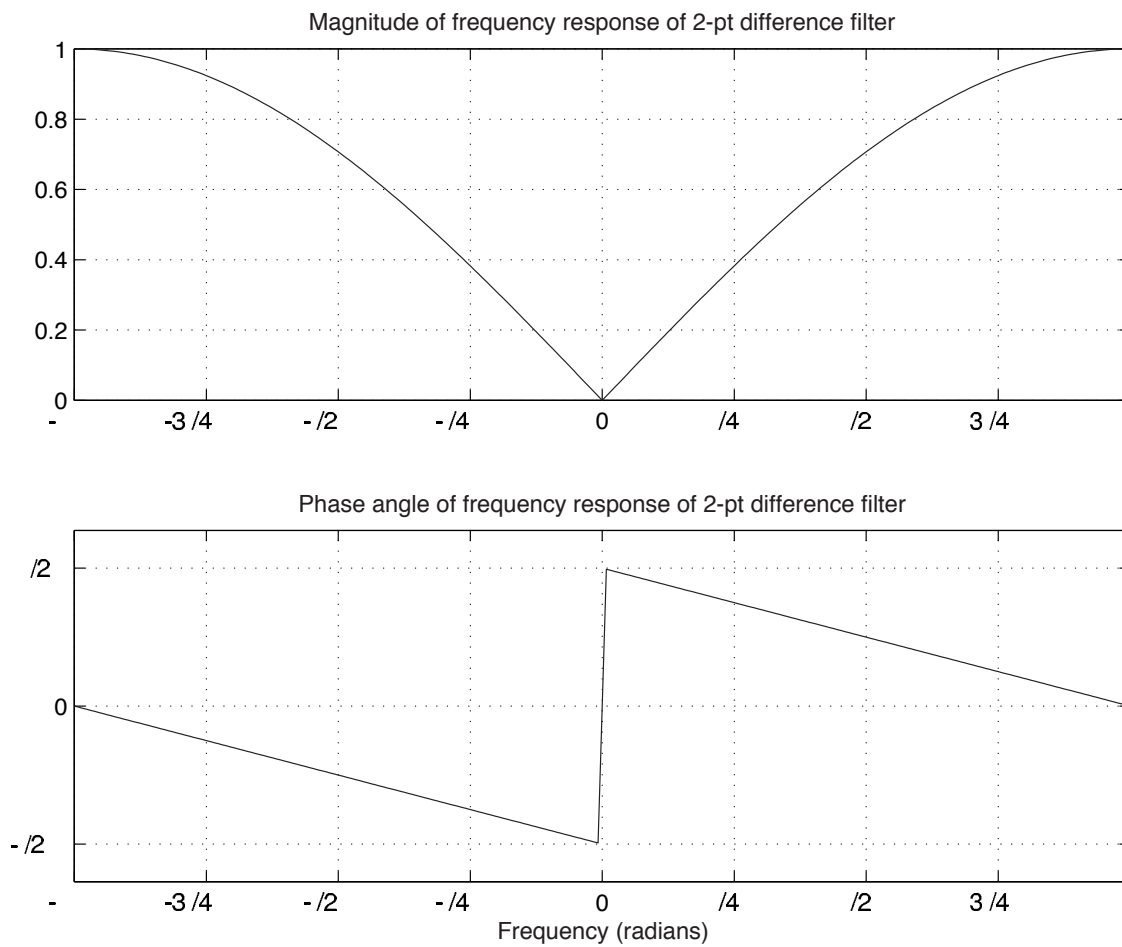


Figure 3: Differencing Filter

Problem 3: Return of the Labs: DSP First Lab 5

Items to be turned in:

- (a) Plots and answers to questions specified in C.5.2.1.
- (b) Plots and answers to questions specified in C.5.3.2.
- (c) Demonstrate linearity and time-invariance of filter (C.5.3.3 and C.5.3.4).
- (d) Plots and answers to questions specified in C.5.3.5.

SOLUTION :

(a) C.5.2.1 Frequency Response of the 3-Point Averager

(1) *The impulse response of a three-point averager is simply:*

$$h[k] = \frac{1}{3}\delta[k] + \frac{1}{3}\delta[k-1] + \frac{1}{3}\delta[k-2]$$

The frequency response can be calculated as follows:

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^M h[k]e^{-j\hat{\omega}k} \\ &= \frac{1}{3}e^{-j\hat{\omega}0} + \frac{1}{3}e^{-j\hat{\omega}1} + \frac{1}{3}e^{-j\hat{\omega}2} \\ &= \frac{1}{3} + \frac{1}{3}e^{-j\hat{\omega}} + \frac{1}{3}e^{-j\hat{\omega}2} \\ &= \frac{1}{3}e^{-j\hat{\omega}} \left(e^{-j\hat{\omega}} + 1 + e^{-j\hat{\omega}} \right) \\ &= \frac{2 \cos(\hat{\omega}) + 1}{3} e^{-j\hat{\omega}}\end{aligned}$$

(2) The following code computes this frequency response directly in MATLAB:

```
function 3ptlowpass()

w = -pi:pi/128:pi;
h = freqz([1/3 1/3 1/3],1,w);

subplot(2,1,1)
plot(w,abs(h))
axis([-pi pi 0 1])
set(gca,'XTick',[-pi:pi/4:pi]);
set(gca,'XTickLabel','|-3/4|-/2|-/4|0|/4|/2|3/4|');
%set(get(gca,'XTickLabel'))
grid
title('Magnitude of frequency response of 3-pt averaging filter')

subplot(2,1,2)
plot(w,angle(h))
axis([-pi pi -pi pi])
set(gca,'XTick',[-pi:pi/4:pi]);
set(gca,'XTickLabel','|-3/4|-/2|-/4|0|/4|/2|3/4|');
set(gca,'YTick',[-pi:pi/2:pi]);
set(gca,'YTickLabel','|-/2|0|/2|');
grid
title('Phase angle of frequency response of 3-pt averaging filter')
xlabel('Frequency (radians)')
```

The following plots are generated:

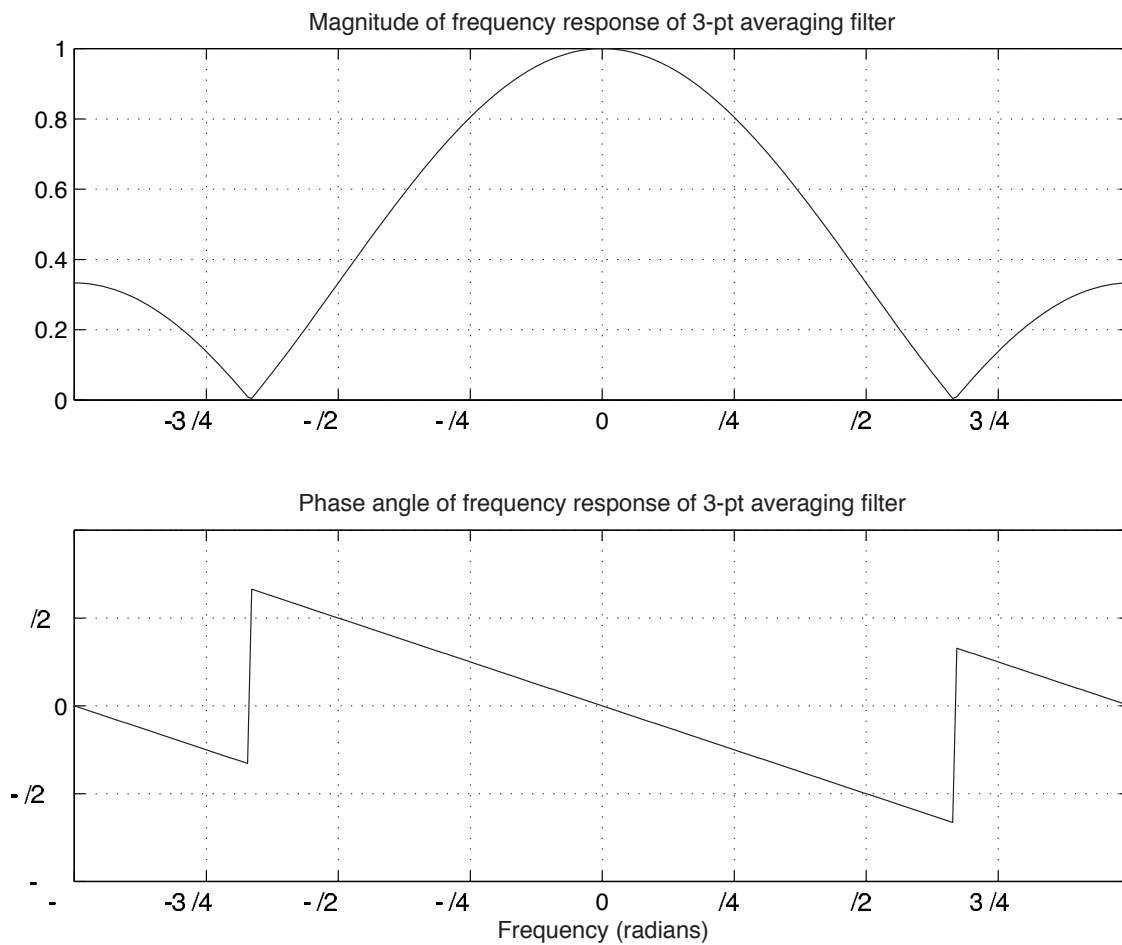


Figure 4: Averaging Filter (again)

(3) The magnitude and phase plots generated computing $H(\hat{\omega})$ directly and using `freqz` to evaluate the response are identical.

(b) C.5.3.2 First-Difference Filter

(1) $x[n]$ and $y[n]$ are not the same length because the filtering is performed via convolution, resulting in a signal length that is the sum of the individual signal lengths minus one ($N + M - 1$).

(2)

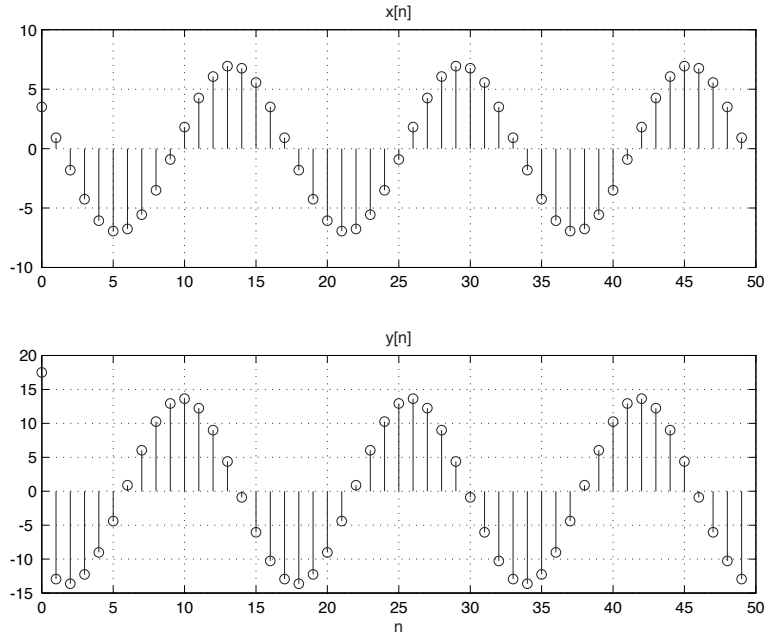


Figure 5: First Order Differencing (Time Domain)

- (3) From the plot, it appears that the amplitude of $x[n] = 7$. The phase is approximately 3 samples which gives us $3(0.125\pi) \approx \frac{\pi}{3}$.
- (4) The filter takes the difference between adjacent samples. Since $x[n]$ is zero for $n < 0$, $y[0]$ is the result of a much greater difference than other values of $y[n]$.
- (5) The frequency remains the same, $\hat{\omega} = 0.125\pi$. The amplitude is approximately 14, and the phase is approximately 6 samples, which gives us $6(0.125\pi) = \frac{3}{4}\pi$.
- (6) At the input frequency $\hat{\omega} = 0.125\pi$, the filter seems to do the following:

$$\text{Amplitude} \approx \frac{14}{7} = 2$$

$$\text{Phase} \approx \frac{3\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$$

(7)

$$\begin{aligned}
 \mathcal{H}(\hat{\omega}) &= 5e^{-j\hat{\omega}0} - 5e^{-j\hat{\omega}1} \\
 &= 5e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) \\
 &= 5e^{-j\hat{\omega}/2}2j \sin(\hat{\omega}/2) \\
 &= 10e^{-j(\hat{\omega}-\pi)/2} \sin(\hat{\omega}/2) \\
 |\mathcal{H}(0.125\pi)| &= 10 \sin(0.125\pi/2) \\
 &\approx 1.951 \\
 \angle\mathcal{H}(0.125\pi) &= -(0.125\pi - \pi)/2 \\
 &= \frac{7}{16}\pi
 \end{aligned}$$

(c) C.5.3.3 Linearity and time invariance of the Filter

(1)

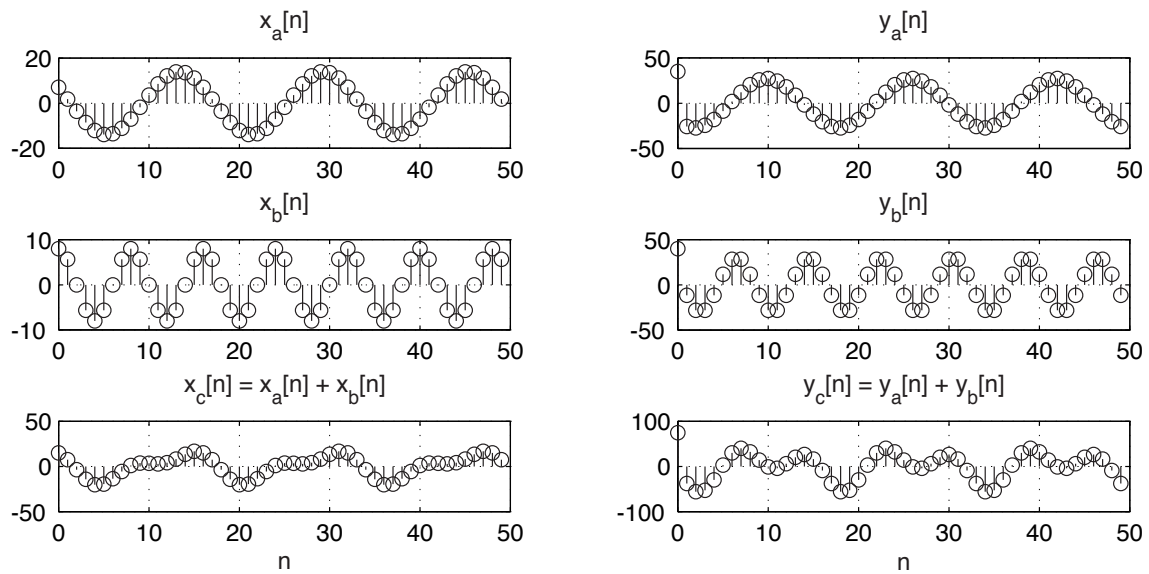


Figure 6: Checking Linearity

From the plot, the amplitude of $x_a[n]$ is approximately 14 and the amplitude of $y_a[n]$ is approximately 28. The phase of $x_a[n]$ is $\frac{\pi}{3}$ and the phase of $y_a[n]$ is about 6 samples, which is $6(0.125\pi) = \frac{3}{4}\pi$. Thus relative change in amplitude is still approximately 2, and the relative change in phase is still $\frac{5}{12}\pi \approx \frac{7}{16}\pi$.

- (2) The amplitude of $x_b[n]$ is 8 and its phase is 0. The amplitude of $y_b[n]$ is approximately 32 and its phase is about 1.5 samples, which is $1.5(0.25\pi) = \frac{3}{8}\pi$. Thus the relative change in amplitude is 4, and the relative change in phase is $\frac{3}{8}\pi$.
- (3) $y_c[n]$ is identical to $y_a[n] + y_b[n]$, demonstrating that the filter is indeed linear.

(4) It is clear from the plot that $y[n]$ must also be shifted to the right by 3 samples to be equal to $y_s[n]$, which demonstrates that the filter is time-invariant.

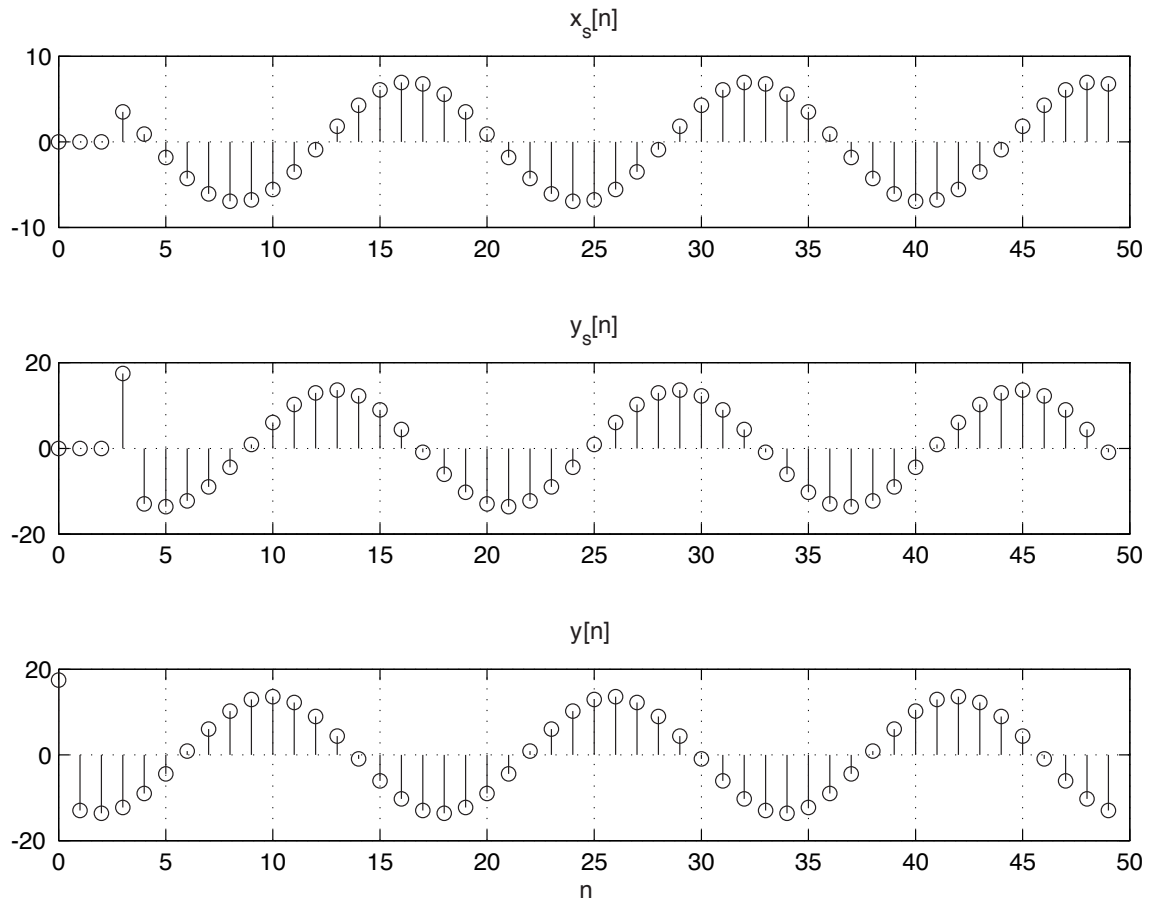


Figure 7: Checking Shift Invariance

(d) Cascading Two-Systems

(1)

(2)

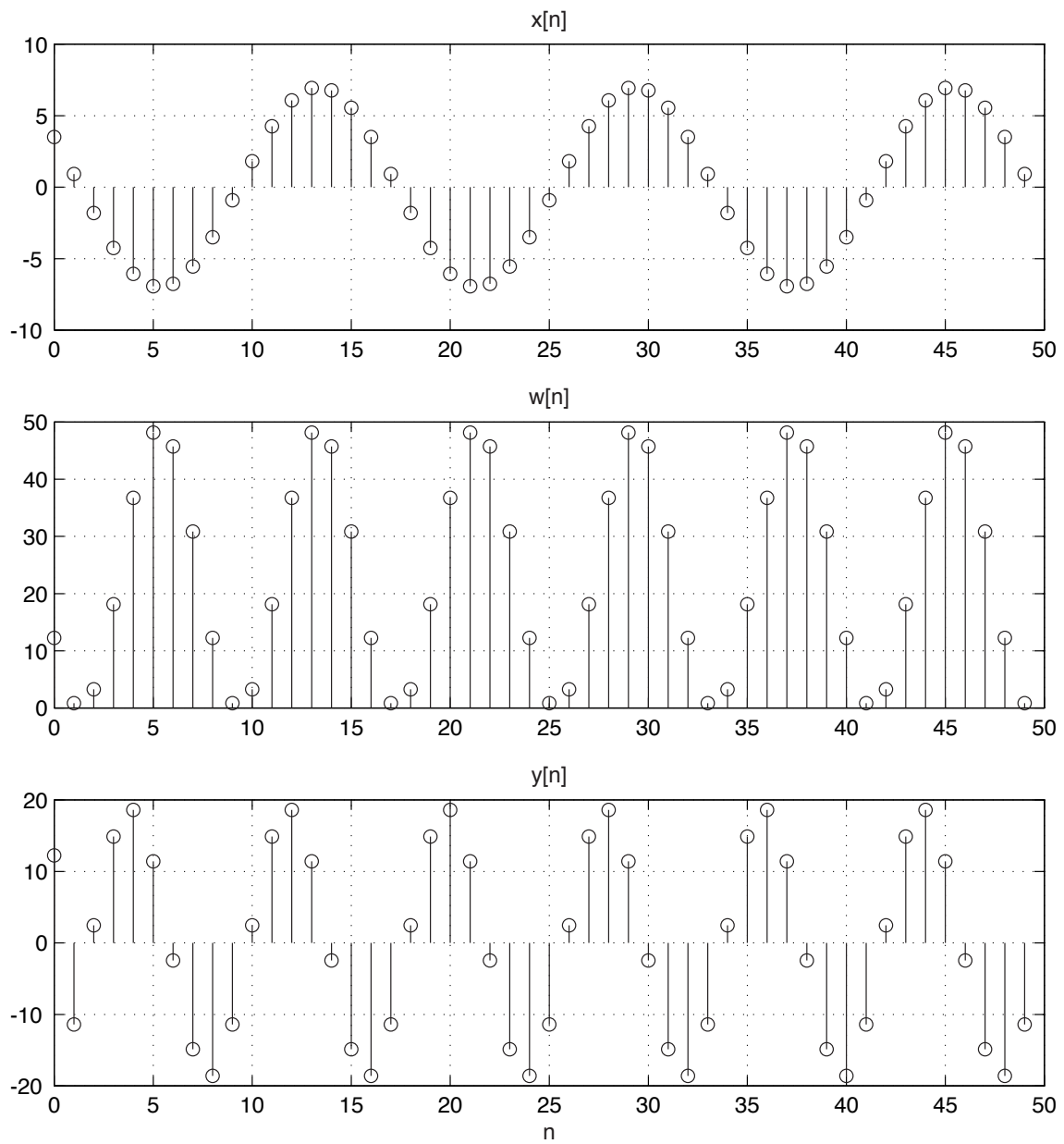


Figure 8: Cascading (Time Domain)

(3) Recall that $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$. From this equation, we see that the squaring operation doubles the frequency and adds a constant (0 frequency) term as well. (From this equation you can also see that squaring doubles the phase as well.) This frequency doubling is apparent in the plot of $w[n]$. Our input $x[n]$ has a frequency of $0.125\pi = \frac{\pi}{8}$. The squaring process doubles that frequency to $0.25\pi = \frac{\pi}{4}$, and adds a constant term (0 frequency), and squares the magnitude as well.

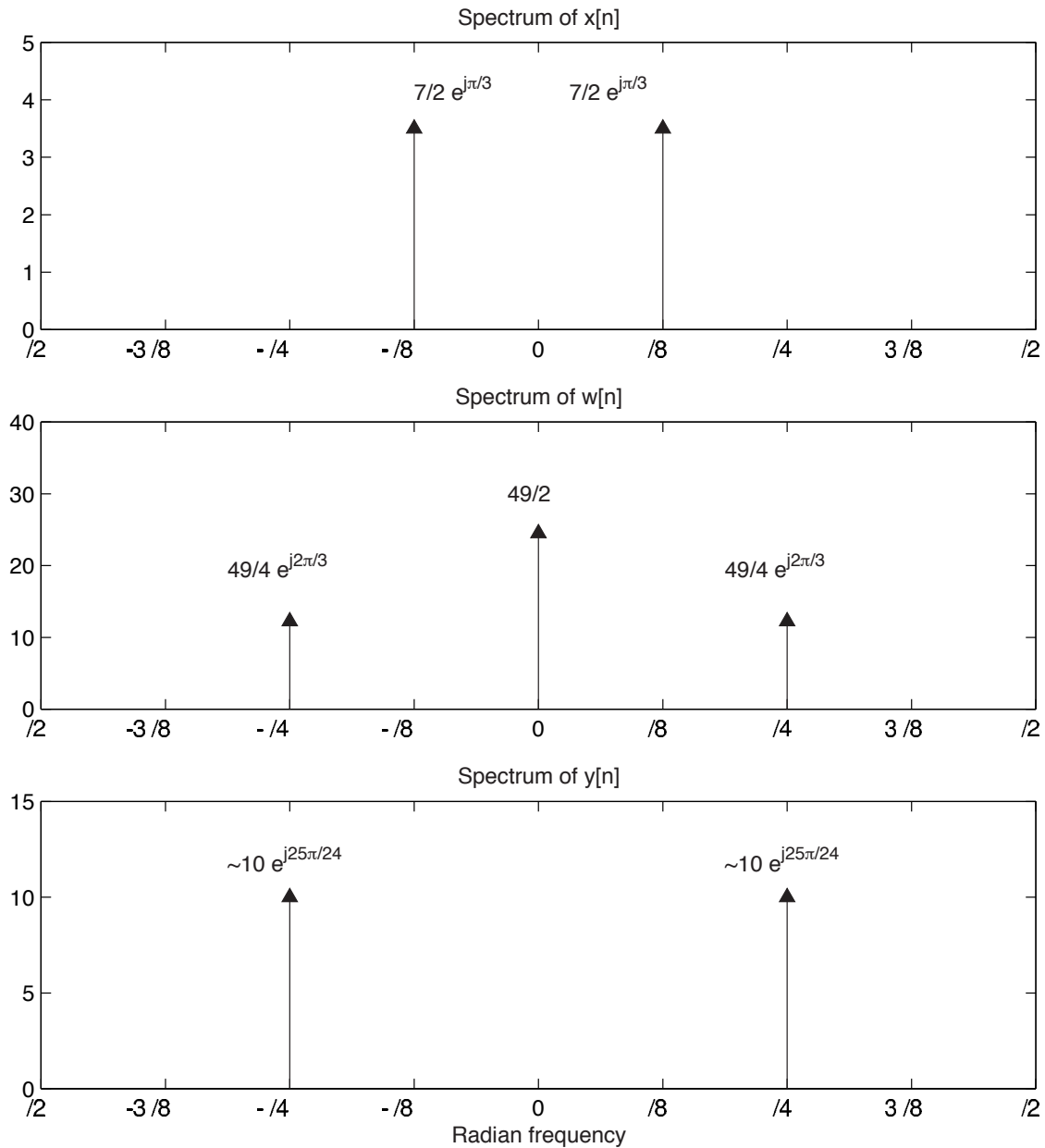


Figure 9: Cascading (Frequency Domain)

- (4) Yes, as can be seen in figure 8 the period of $x[n]$ is around 15. The period of $w[n]$ is around 8.
- (5) At the frequencies of $w[n]$, the first-difference filter has a magnitude response of:

$$|\mathcal{H}(0)| = 2 \sin(0/2) = 0$$

$$|\mathcal{H}(0.25\pi)| = 2 \sin(0.25\pi/2) \approx 0.7654$$

So $y[n]$ removes the constant (0 frequency term) and has a magnitude of $(0.7654)(49/2) \approx 20$ due to the frequency of 0.25π in $w[n]$.

As we showed above, the first-difference filter has a linear phase of $-(\hat{\omega} - \pi)/2$. Since our input signal $x[n]$ has a phase of $\frac{\pi}{3}$, the squaring operation doubles it to $\frac{2\pi}{3}$. The first-difference filter then changes the phase to be:

$$\frac{2\pi}{3} - \frac{0.25\pi - \pi}{2} = \frac{25\pi}{24}$$

- (6) Now we replace the first-difference filter with the second-order FIR filter $y_2[n] = w[n] - 2\cos(0.25\pi)w[n-1] + w[n-2]$. There are no frequencies present in the output signal because the filter is designed to have zero magnitude at $\hat{\omega} = \frac{\pi}{4}$. When the input to the filter is $w[n] = (x[n])^2$, all that remains is the constant (0 frequency) term. When $w[n] = e^{j0.25\pi n}$, the resulting output is zero. The spectrum of $y_2[n]$ in this case is simply zero as well.

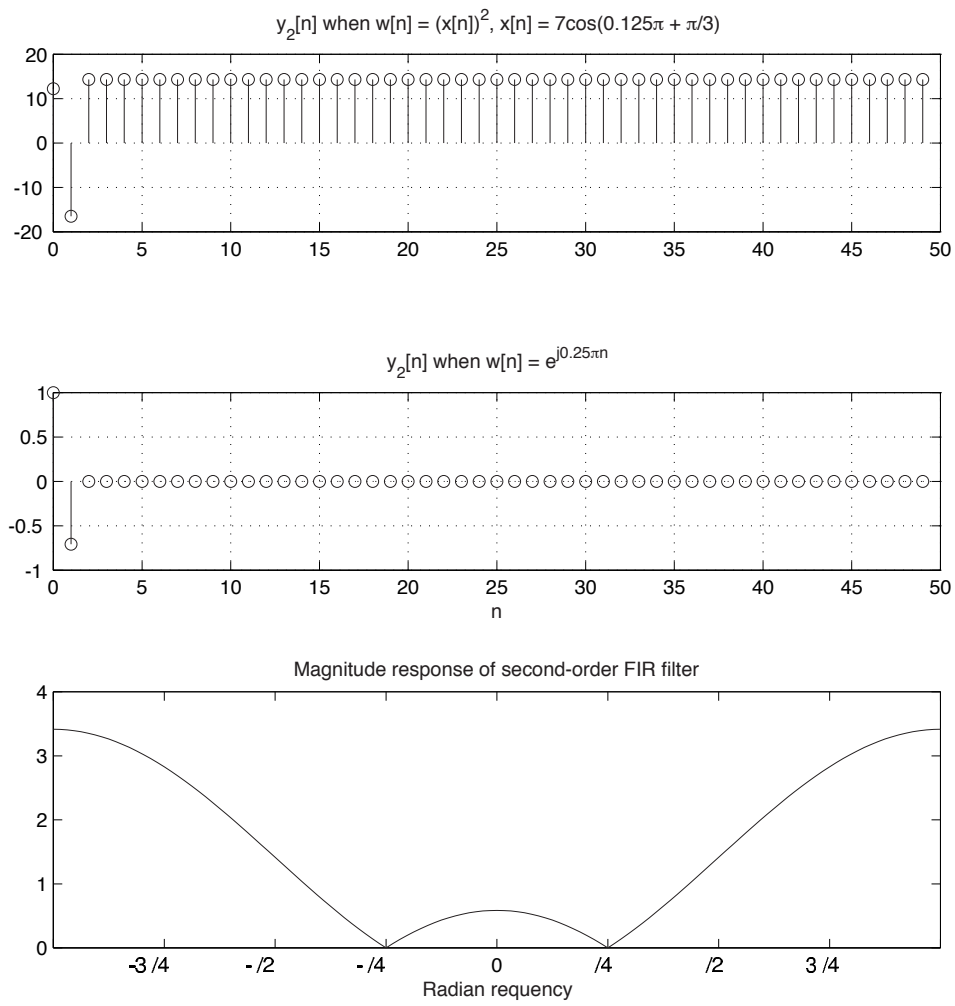


Figure 10: Second Order Differencing

Problem 4: Additional Problem (for MAS.510)

The MATLAB function `zplane` is great for plotting the poles and zeros of a system in the z -plane. Use `zplane` as well as `freqz` to answer the following questions.

- (a) Consider the general N -point FIR averaging filter, where each coefficient b_k is simply $\frac{1}{N}$. Plot the zeros of this system in the z -plane as well as the frequency response (magnitude and phase) for $N = 3, 4, 5,$ and 10 . How does the position of the zeros change? Qualitatively, how does the frequency response change, and how does this relate to the location of the zeros? Does N being an even or odd number have any effect?
- (b) Consider the general N -point FIR difference filter, where each coefficient b_k is simply $\frac{(-1)^k}{N}$. Plot the zeros of this system in the z -plane as well as the frequency response (magnitude and phase) for $N = 3, 4, 5,$ and 10 . How does the position of the zeros change? Qualitatively, how does the frequency response change, and how does this relate to the location of the zeros? Does N being an even or odd number have any effect?

SOLUTION :

(a) The zeros are evenly spaced around the unit circle (except at $z = 1$, making this a low-pass filter). The more zeros that are added, the narrower the pass region of the filter. If N is even, there is a zero at $z = -1$ (maximum frequency), but if N is odd, there is not.

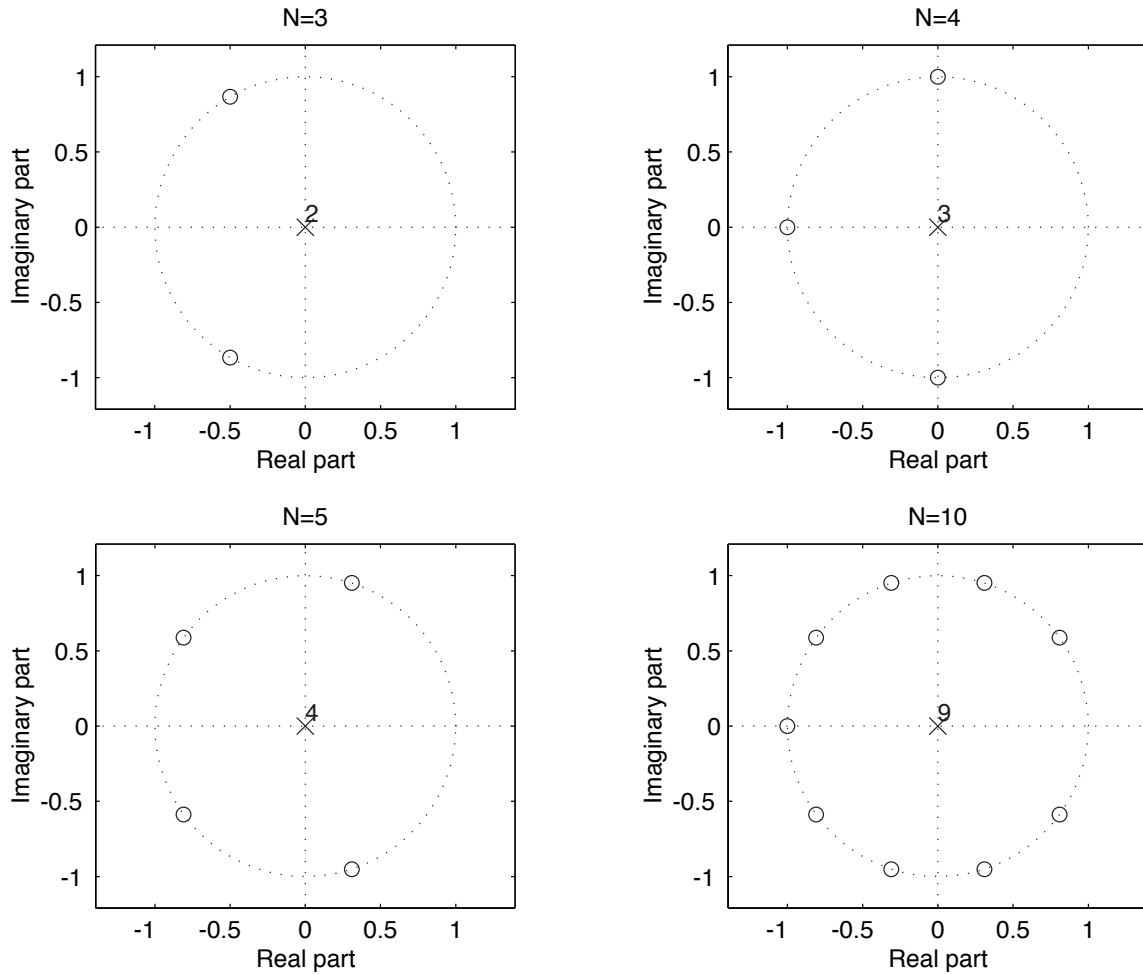


Figure 11: Zeros of Averaging Filters

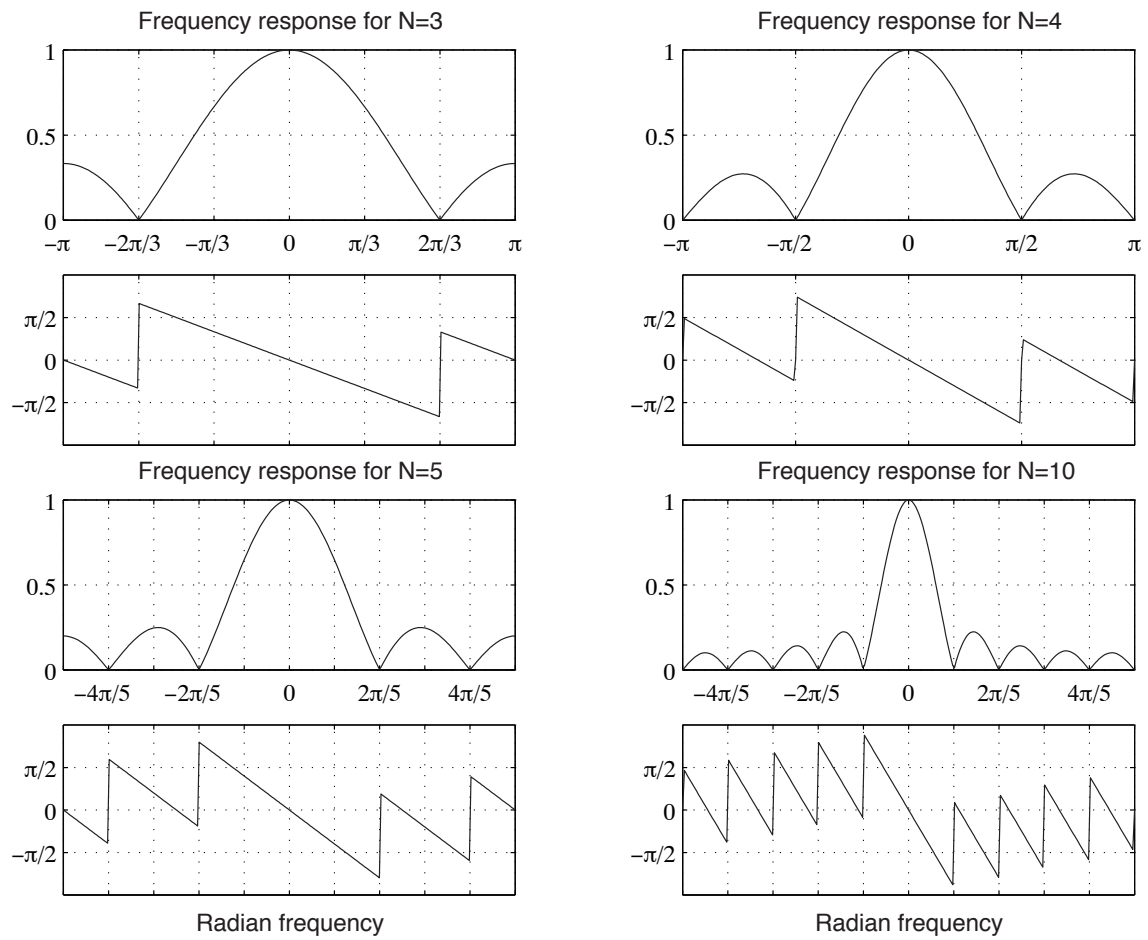


Figure 12: Frequency Response of Averaging Filters

(b) The zeros are evenly spaced around the unit circle (except at $z = -1$, making this a high-pass filter). Again, the more zeros that are added, the narrower the pass region of the filter. If N is even, there is a zero at $z = 1$ (zero frequency), but if N is odd, there is not.

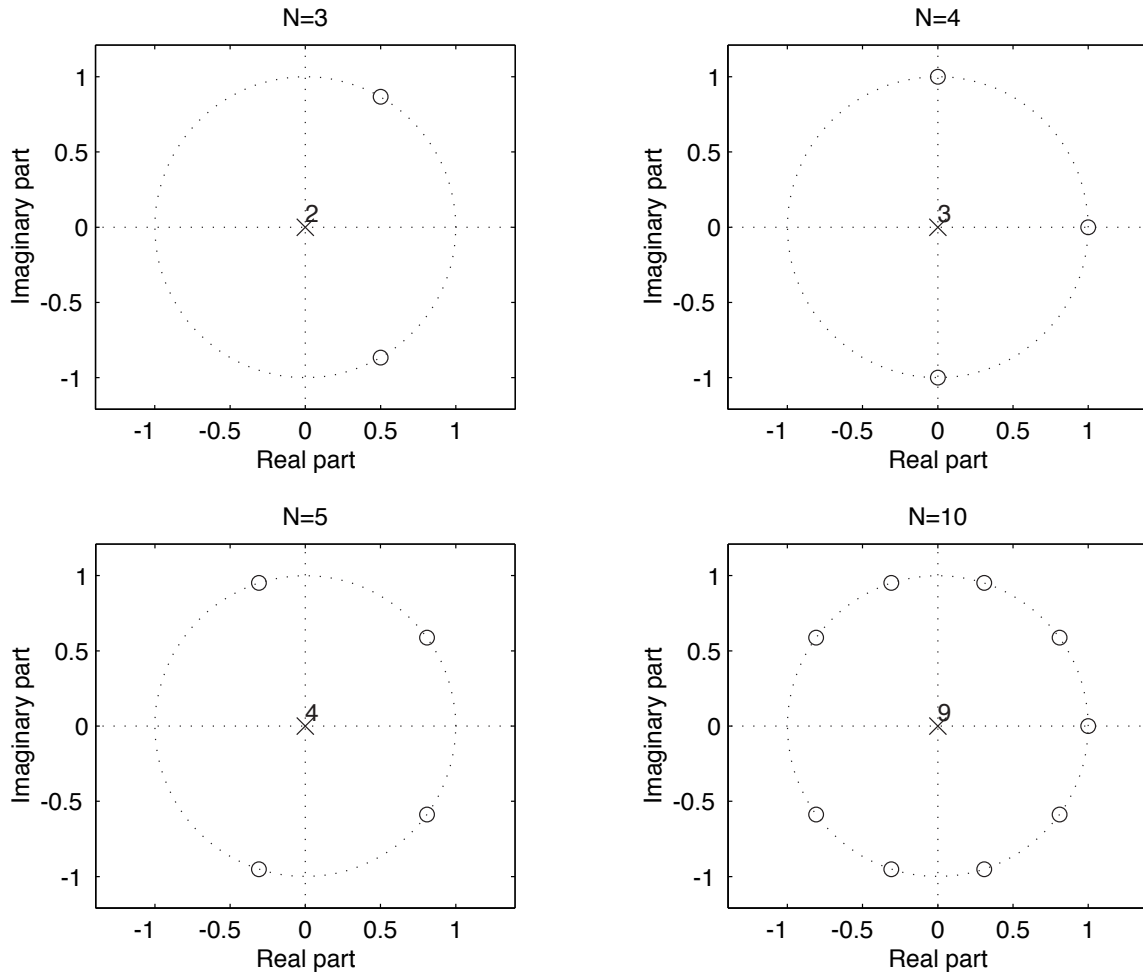


Figure 13: Zeros of Differencing Filters

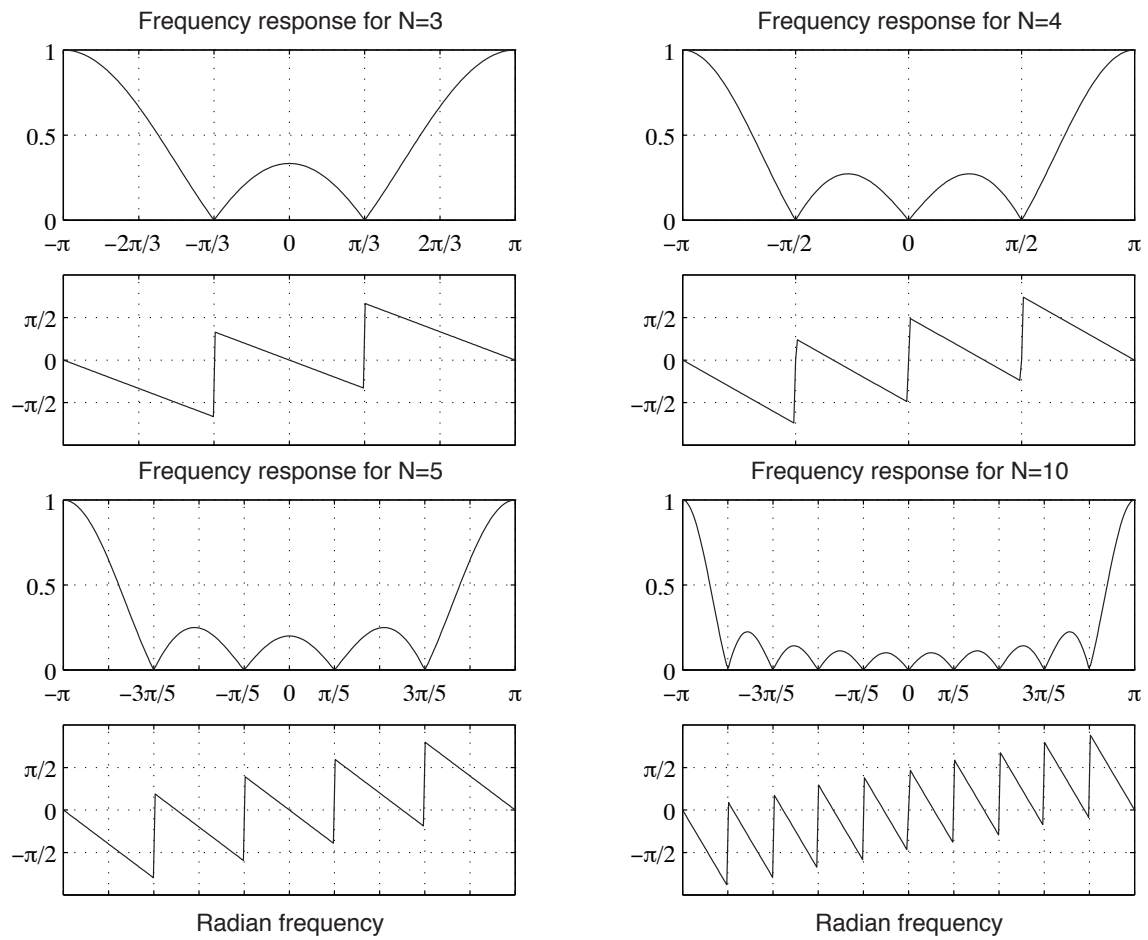


Figure 14: Frequency Response of Averaging Filters