

$$y[n] = \sum_{k=0}^M b_k x[n-k] \quad \text{FIR}$$

$$y[n] = \sum_{k=0}^M h[k] x[n-k] \quad \text{convolution}$$

$$x[n] = Ae^{j\phi} e^{j\hat{\omega}n} \quad \text{Complex exponential input}$$

$$\begin{aligned} &\downarrow \\ y[n] &= \sum_{k=0}^M h[k] Ae^{j\phi} e^{j\hat{\omega}(n-k)} \\ &= \left(\sum_{k=0}^M h[k] e^{-j\hat{\omega}k} \right) Ae^{j\phi} e^{j\hat{\omega}n} \\ &= \mathcal{H}(\hat{\omega}) Ae^{j\phi} e^{j\hat{\omega}n} \end{aligned}$$

let

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^M h[k] e^{j\hat{\omega}k}$$

$\mathcal{H}(\hat{\omega})$ frequency response

Ex. $y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$ FIR

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{k=0}^2 h[k]e^{-j\hat{\omega}k} \\&= h[0]e^{-j\hat{\omega}0} + h[1]e^{-j\hat{\omega}1} + h[2]e^{-j\hat{\omega}2} \\&= \frac{1}{3} + \frac{1}{3}e^{-j\hat{\omega}} + \frac{1}{3}e^{-j\hat{\omega}2} \\&= \frac{1}{3}\left(1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right) \\&= \frac{1}{3}e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + 1 + e^{-j\hat{\omega}}\right) \\&= \frac{1}{3}e^{-j\hat{\omega}}\left(1 + 2\cos\hat{\omega}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \frac{1}{3} e^{-j\hat{\omega}} (1 + 2 \cos \hat{\omega}) \\ &= \frac{1}{3} (1 + 2 \cos \hat{\omega}) (\cos \hat{\omega} - j \sin \hat{\omega})\end{aligned}$$

$$|\mathcal{H}(\hat{\omega})| = \frac{1}{3} |1 + 2 \cos \hat{\omega}|$$

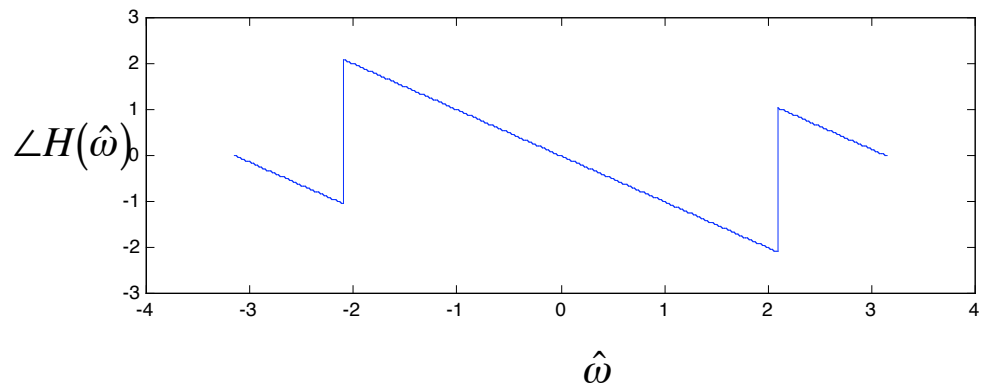
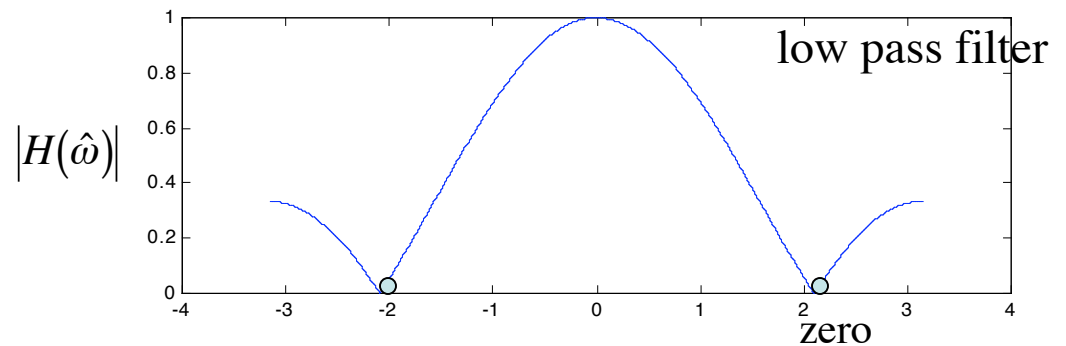
Note: $|\mathcal{H}(\frac{2\pi}{3})| = 0$

$$\angle \mathcal{H}(\hat{\omega}) = -\hat{\omega} \quad \text{linear phase}$$

$$= \begin{cases} -\hat{\omega} & 2\pi/3 \leq \hat{\omega} < 0 \\ -\hat{\omega} + \pi & \pi \leq \hat{\omega} < 2\pi/3 \end{cases}$$

$$\angle \mathcal{H}(-\hat{\omega}) = -\angle \mathcal{H}(\hat{\omega})$$

principal value of phase fn



Normalized vs. actual frequency

$$|\mathcal{H}(\hat{\omega})| = \frac{1}{3} |1 + 2 \cos \hat{\omega}|$$

Note: $|\mathcal{H}(\frac{2\pi}{3})| = 0$

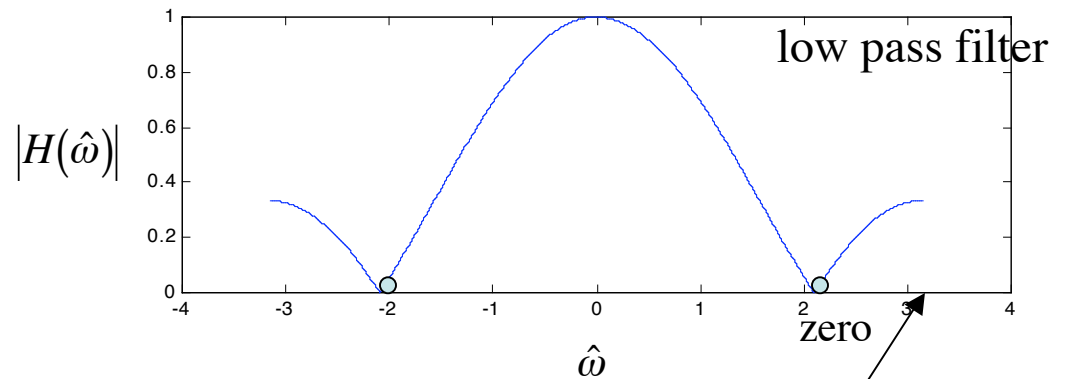
$$\hat{\omega} = \frac{2\pi}{3} \quad \text{normalized frequency}$$

$$f = ? \quad \text{actual frequency (Hz)}$$

$$\hat{f} = \frac{f}{f_s}$$

$$\hat{\omega} = 2\pi\hat{f} = 2\pi \frac{f}{f_s}$$

$$f = \frac{\hat{\omega} f_s}{2\pi}$$



$$-\pi < \hat{\omega} < \pi$$

$$f|_{\hat{\omega}=\pi} = \frac{\pi f_s}{2\pi} = \frac{f_s}{2} \quad \text{Nyquist}$$

$$\hat{\omega} = \frac{2\pi}{3}, f_s = 8000 \text{ Hz}$$

$$f = \frac{\frac{2\pi}{3} 8000}{2\pi} = \frac{8000}{3} \text{ Hz} = 1667 \text{ Hz}$$

Convolution / solving the difference equation

$$x[n] = 3 + 3\cos(0.6\pi n) \quad \text{input}$$

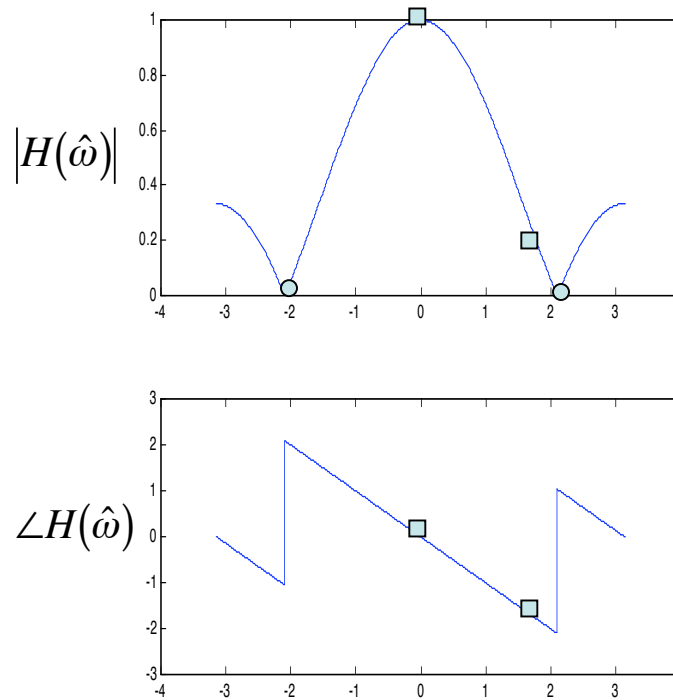
$$y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2] \quad \text{FIR filter}$$

sample domain

$$\begin{aligned} y[n] &= 1 + \cos(0.6\pi n) + 1 + \cos(0.6\pi(n-1)) + 1 + \cos(0.6\pi(n-2)) \\ &= 3 + \cos(0.6\pi n) \\ &\quad + \cos(0.6\pi)\cos(0.6\pi n) + \sin(0.6\pi)\sin(0.6\pi n) \\ &\quad + \cos(1.2\pi)\cos(0.6\pi n) + \sin(1.2\pi)\sin(0.6\pi n) \\ &\quad \vdots \\ &= 3 + 0.382\cos(0.6\pi n - 0.6\pi) \end{aligned}$$

Multiplication in frequency domain

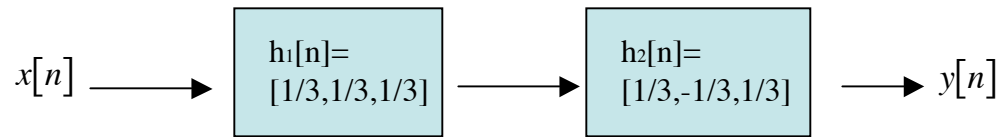
$$x[n] = 3 + 3\cos(0.6\pi n)$$



closer $\hat{\omega}$
to a zero,
the smaller the
output.

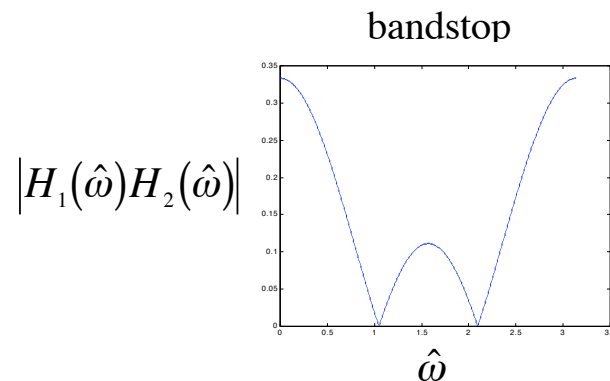
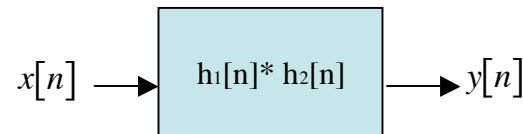
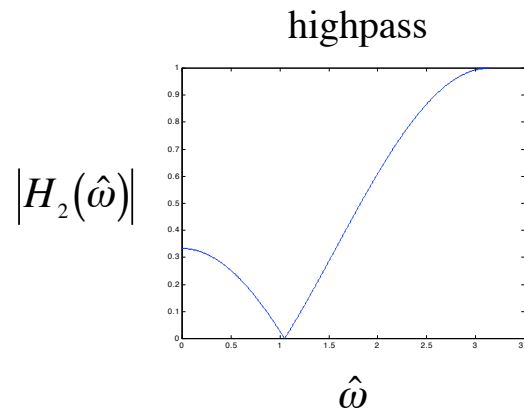
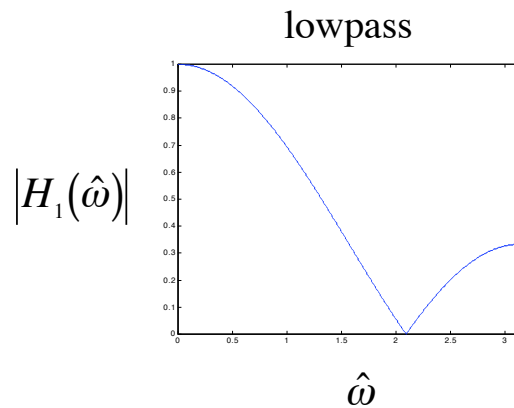
$$\begin{aligned} |\mathcal{H}(\hat{\omega})| &= \frac{1}{3}(1 + 2\cos\hat{\omega}) & \hat{\omega} & \quad |\mathcal{H}(0)| = 1 \quad |\mathcal{H}(0.6\pi)| = 0.127 \\ \angle \mathcal{H}(\hat{\omega}) &= -\hat{\omega} & \angle \mathcal{H}(0) &= 0 \quad \angle \mathcal{H}(0.6\pi) = -0.6\pi \end{aligned}$$

$$\begin{aligned} y[n] &= 3(1) + 3(0.127)\cos(0.6\pi n - 0.6\pi) \\ &= 3 + 0.382\cos(0.6\pi n - 0.6\pi) \end{aligned}$$



$$h_1[n] = [1/3, 1/3, 1/3]$$

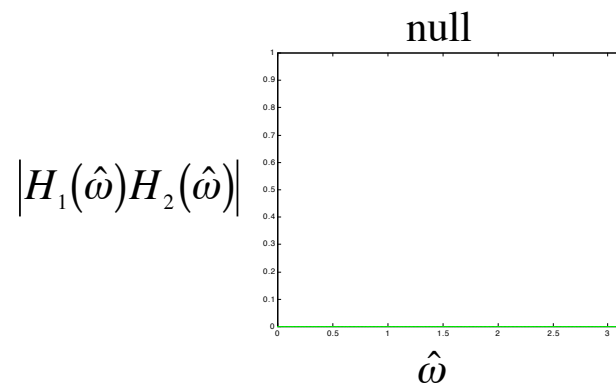
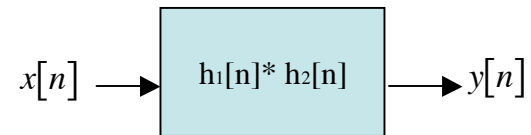
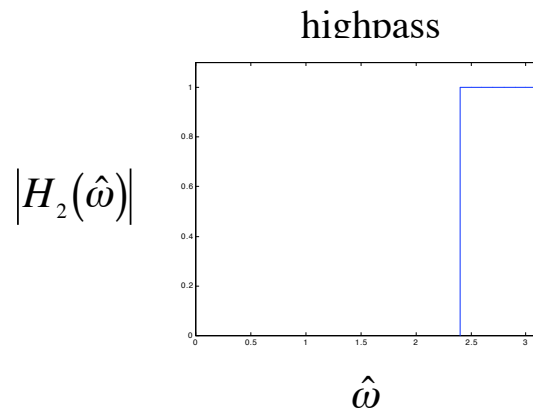
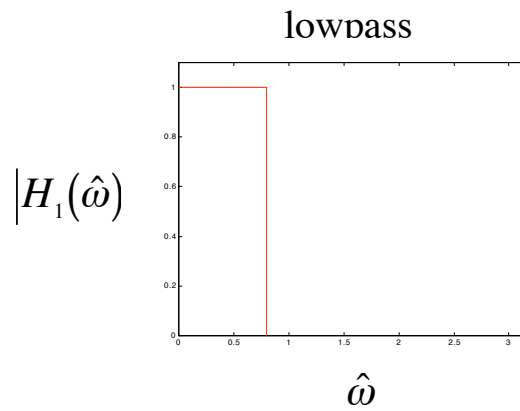
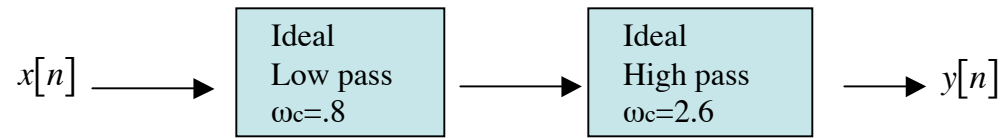
$$h_2[n] = [1/3, -1/3, 1/3]$$



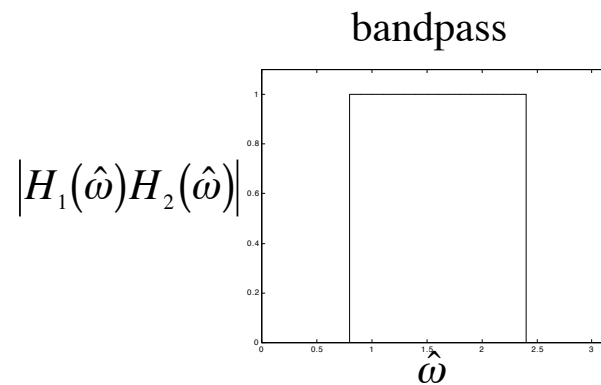
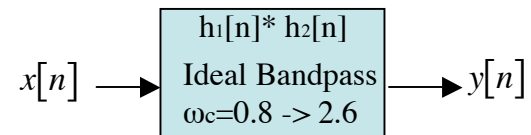
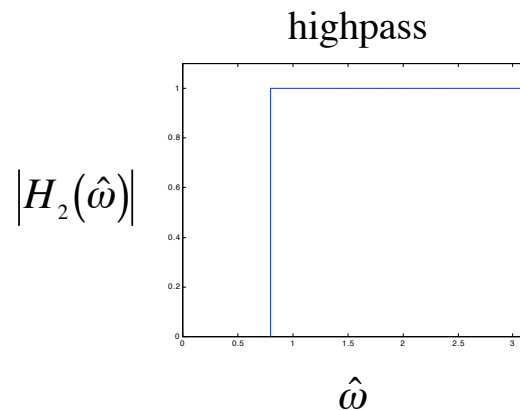
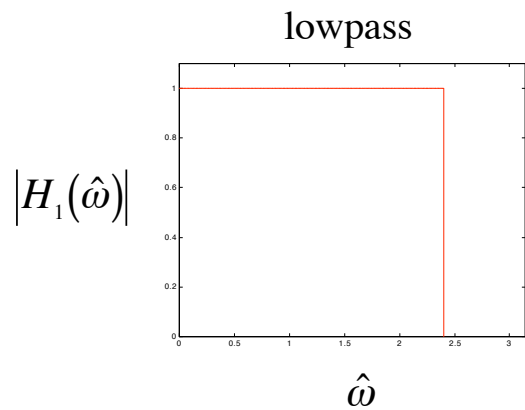
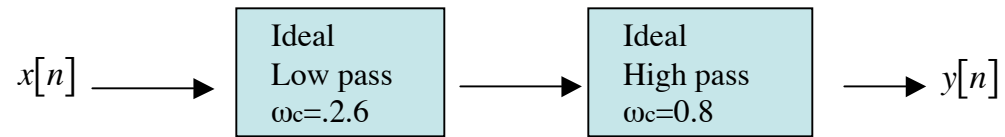
$$h_1[n] * h_2[n] = [1/9, 0, 1/9, 0, 1/9]$$

$$\mathcal{H}(\hat{\omega}) = \mathcal{H}_1(\hat{\omega}) \cdot \mathcal{H}_2(\hat{\omega})$$

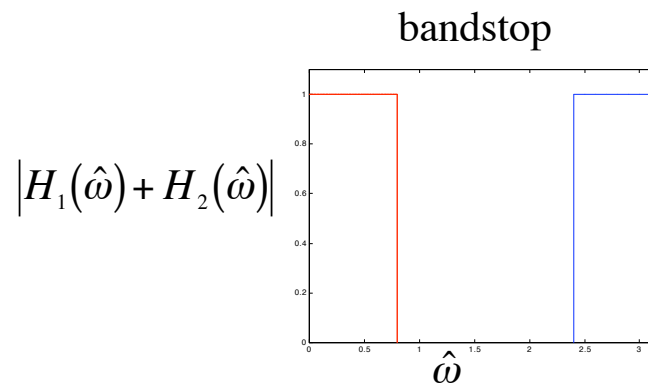
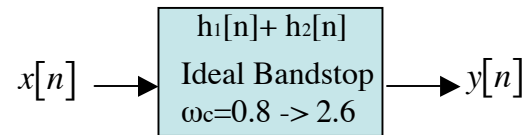
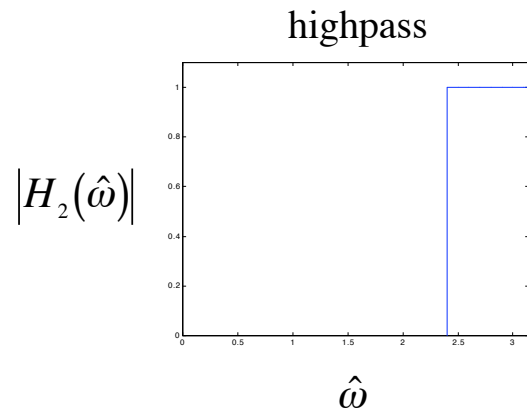
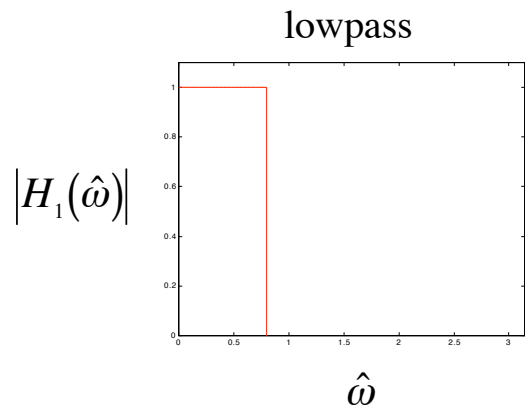
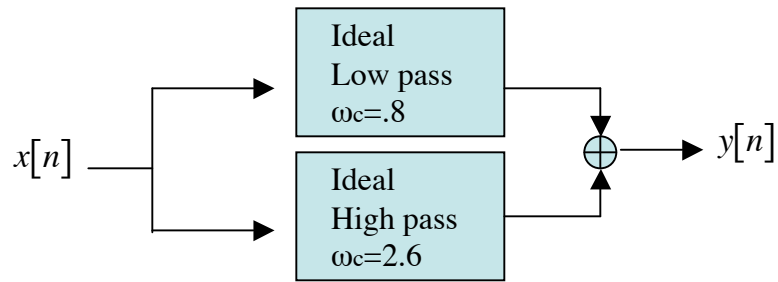
sample domain convolution
= freq domain multiplication
(and visa versa)



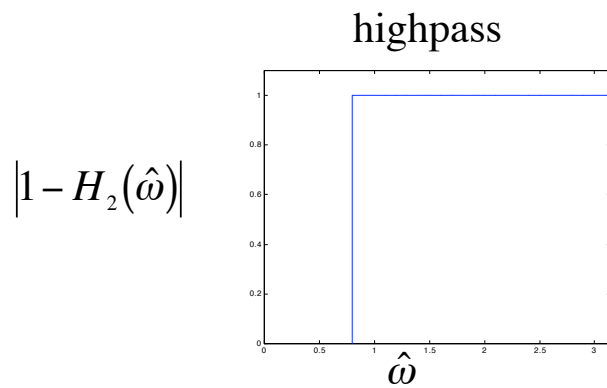
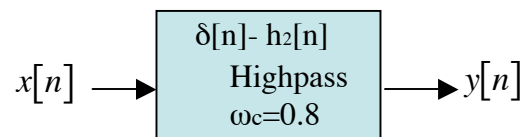
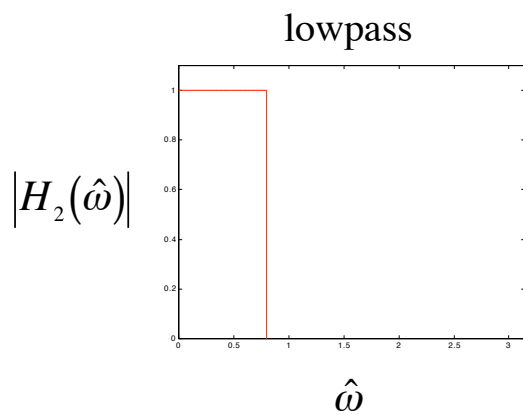
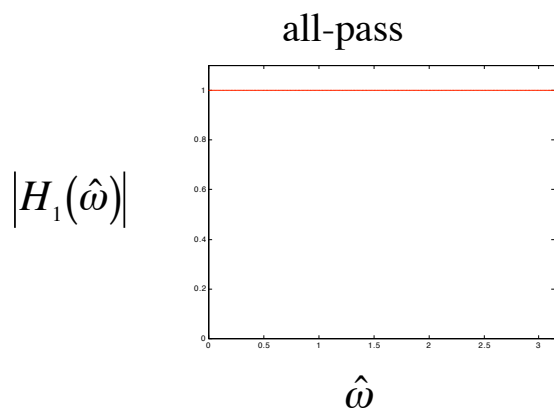
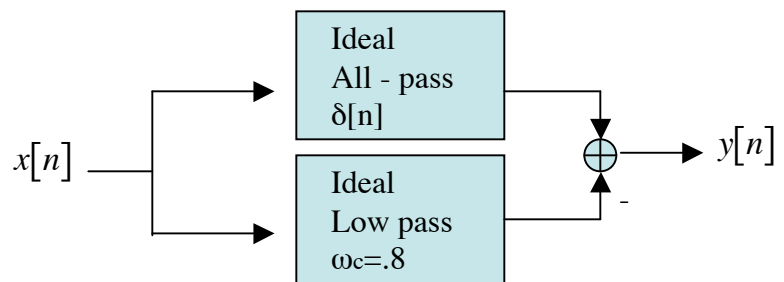
passbands don't overlap



passbands overlap

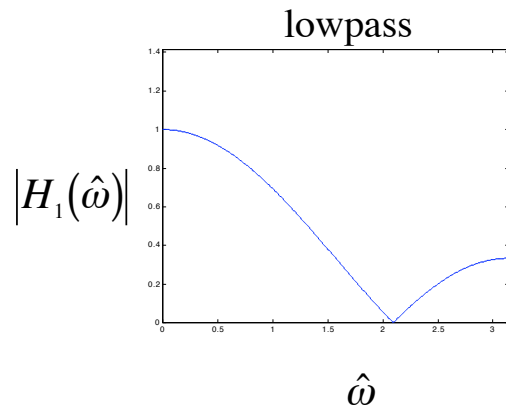


passbands don't overlap

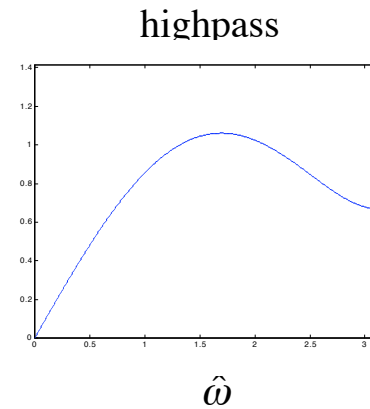


$$h_1[n] = [1/3, 1/3, 1/3]$$

$$h_2[n] = \delta[n] - h_1[n] = [2/3, -1/3, -1/3]$$

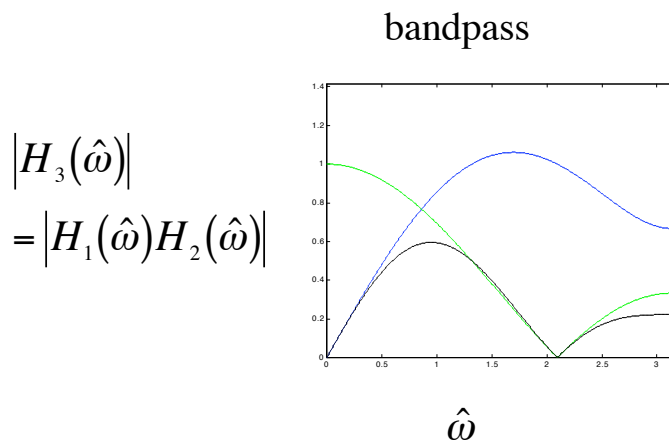


$$|H_2(\hat{\omega})| = |1 - H_1(\hat{\omega})|$$

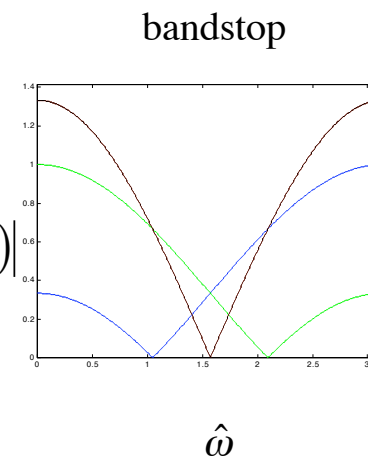


$$h_3[n] = h_1[n] * h_2[n] \\ = [2/9, 1/9, 0, -2/9, -1/9]$$

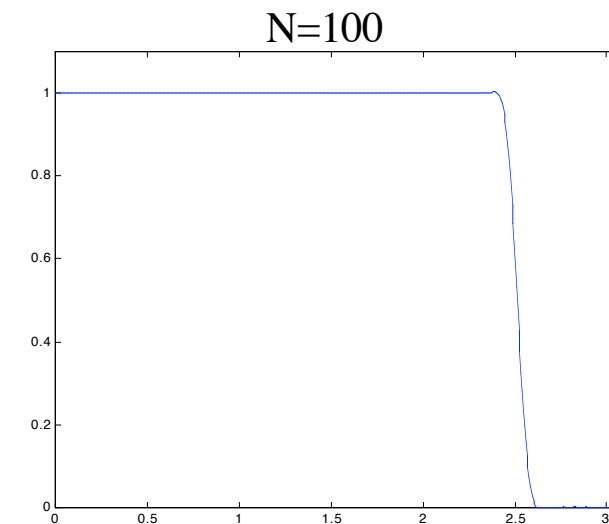
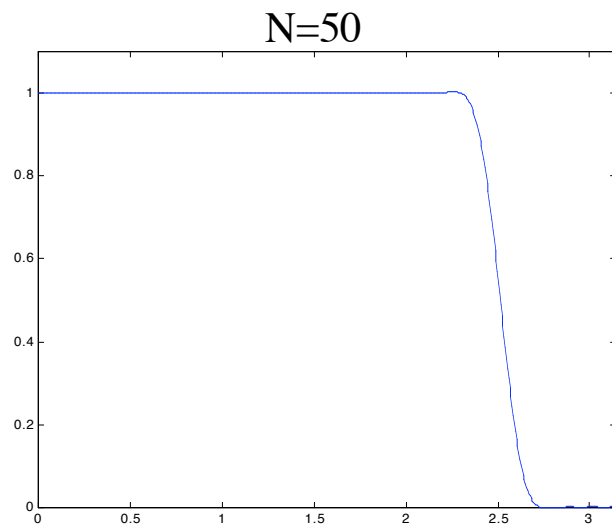
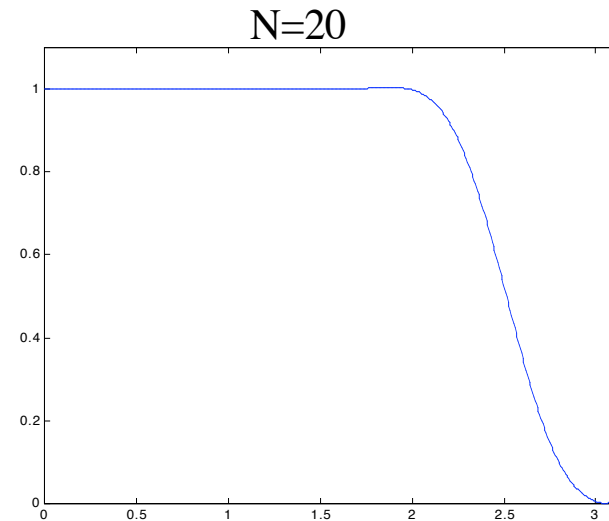
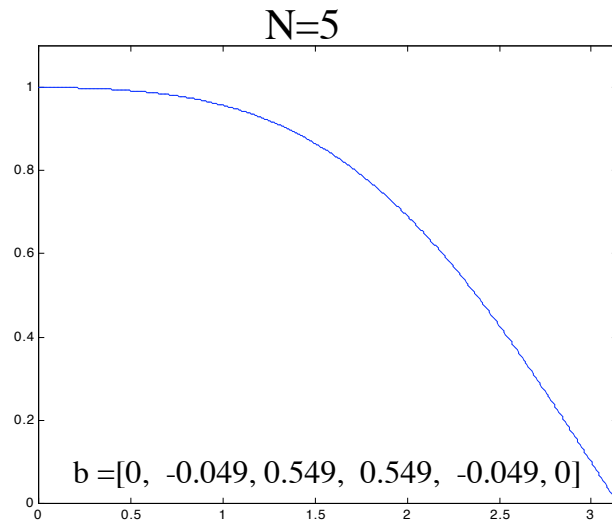
$$h_4[n] = h_1[n] + [1/3 \ -1/3 \ 1/3] \\ = [2/3 \ 0 \ 2/3]$$



$$|H_4(\hat{\omega})| = |H_4(\hat{\omega})| + |H_5(\hat{\omega})|$$



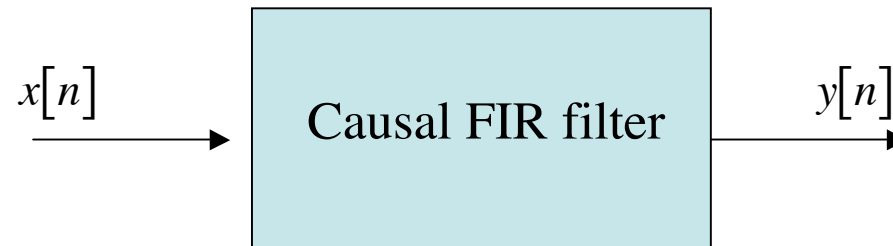
FIR low-pass filter response vs. number of taps N



More coefficients (“taps”), can achieve more ideal response;
Longer to compute, more memory, greater phase shift

Note: These aren’t L-point averagers,
fir1(), Hammingwindow, Wn=0.8

Causal FIR Filter



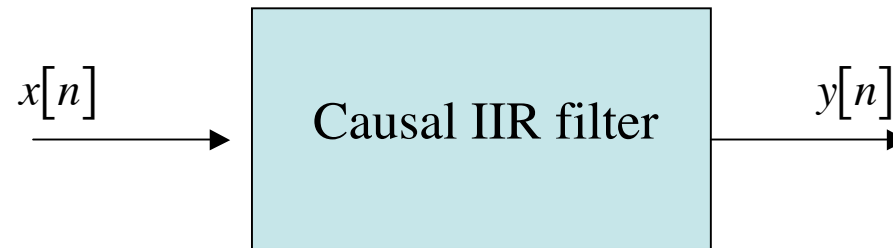
In an FIR filter, the output y at each sample n is a weighted sum of the present input, $x[n]$, and past inputs, $x[n-1]$, $x[n-2]$, ..., $x[n-M]$.

$$y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

Causal IIR filter

Infinite Impulse Response



For an IIR filter, the output y at each sample n is a weighted sum of the present input, $x[n]$, past inputs, $x[n-1]$, $x[n-2]$, ..., $x[n-M]$, and past outputs, $y[n-1]$, $y[n-2]$, ..., $y[n-N]$.

$$y[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M] \\ + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N]$$

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k] \quad N \geq M \quad \text{proper system}$$

IIR Impulse Response

$$x[n] = 1\delta[n]$$

$$y[n] = \frac{1}{2}y[n-1] + 2x[n]$$

To calculate output at step n , need previous outputs.

At step $n=0$, assume initial rest conditions. $x[n]=0$, $y[n]=0$ for $n<0$

$$\begin{aligned}y[0] &= \frac{1}{2}y[0-1] + 2x[0] \\&= \frac{1}{2} \cdot 0 + 2 \cdot 1 \\&= 2\end{aligned}$$

$$\begin{aligned}y[1] &= \frac{1}{2}y[1-1] + \frac{1}{4}x[1] \\&= \frac{1}{2}y[0] + \frac{1}{4}x[1] \\&= \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 0 \\&= 1\end{aligned}$$

for $n>0$

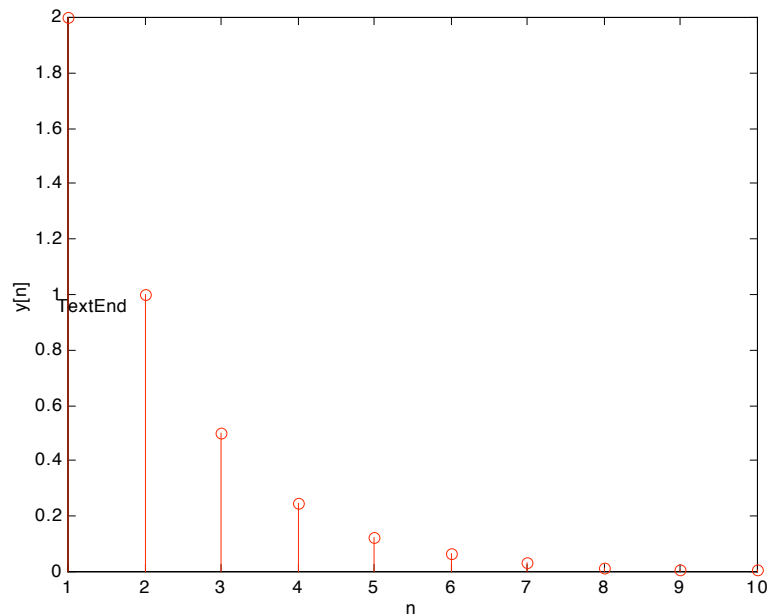
$$\begin{aligned}y[n] &= \frac{1}{2}y[n-1] + 2\cancel{x[n]}^0 \\&= \frac{1}{2}y[n-1]\end{aligned}$$

$$n = [-1 \quad -2 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots]$$

$$x[n] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

$$y[n] = \frac{1}{2} y[n-1] + 2x[n]$$

$$y[n] = \begin{bmatrix} 0 & 0 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots \cdot 2 \cdot \frac{1}{2}^n \end{bmatrix} = h[n] \quad \text{impulse response}$$



decays exponentially,
goes on forever
“infinite impulse response”

IIR impulse response

$$y[n] = \sum_{l=0}^N a_l x[n-l] + \sum_{k=0}^M b_k x[n-k]$$

IIR filter

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Delta function

↓

$$y[n] \Big|_{x=\delta[n]} = h[n] \neq \sum_{k=0}^M b_k \delta[n-k]$$

impulse response

Can't read filter coefficients
off of impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Convolution sum:
LTI systems: FIR/IIR

IIR convolution

$$x[n] = x[k]\delta[n - k]$$

We can decompose $x[n]$
into scaled delayed impulses.
Here assume finite length $x[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Convolution sum:
LTI: FIR / IIR

$$y[n] = x[0]h[n] + x[1]h[n - 1] + x[2]h[n - 2] + \dots$$

Each impulse has a scaled impulse response.
The final output is a sum of those scaled and delayed
impulse responses. Now the impulse responses are
Infinite in length.

IIR convolution

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-2]$$

$$h[n] = \begin{bmatrix} 0 & 0 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots 2 \cdot \frac{1}{2}^n \\ n=-2 & -1 & 0 & 1 & 2 & 3 & 4 & \dots \end{bmatrix}$$

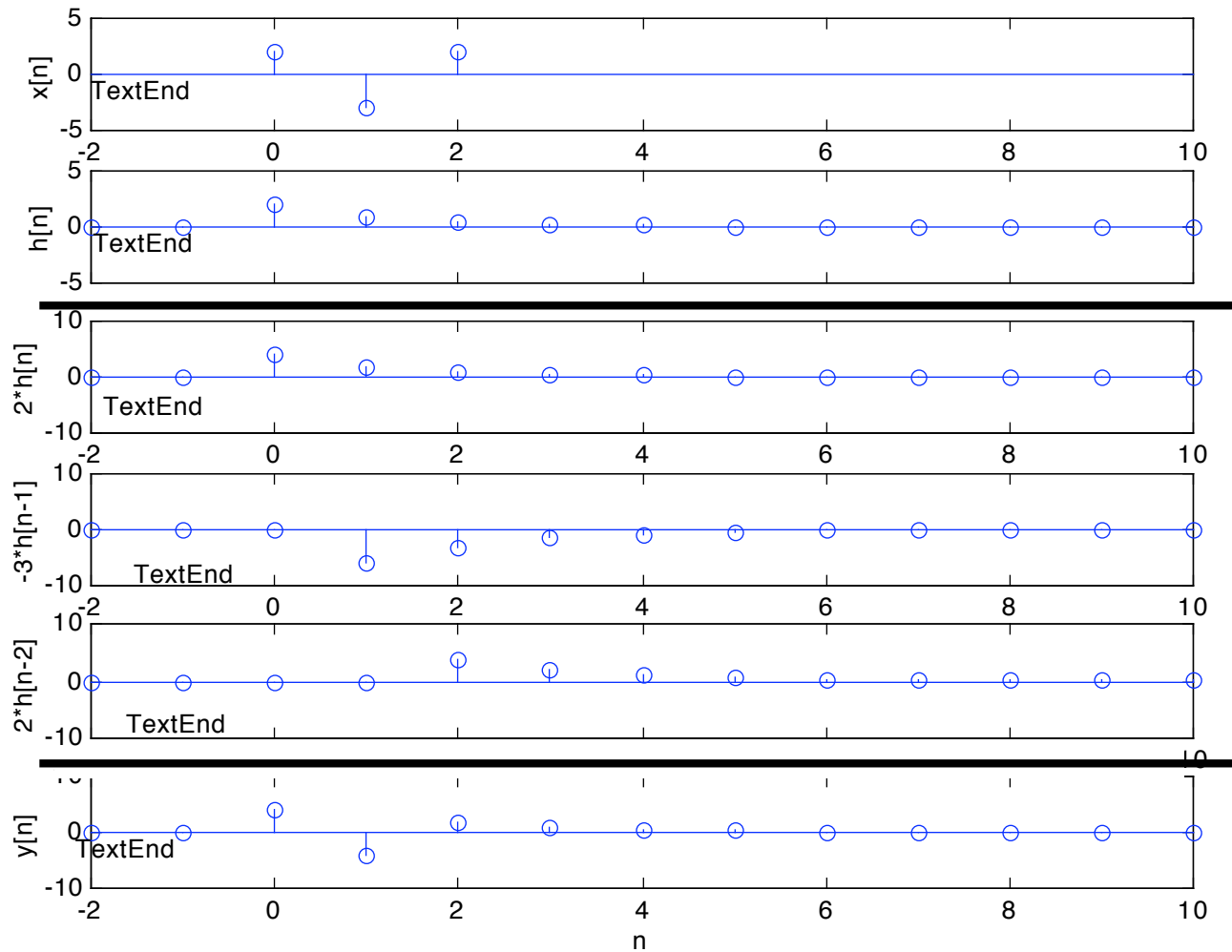
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Each impulse has a scaled impulse response. The final output is a sum of those scaled and delayed impulse responses.

$$y[n] = 2 \cdot h[n] - 3 \cdot h[n-1] + 2 \cdot h[n-2]$$

$$= \begin{bmatrix} 0 & 0 & 4 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots & 2 \cdot \frac{1}{2}^{n-1} \\ 0 & 0 & 0 & -6 & -3 & \frac{-3}{2} & \frac{-3}{4} & \frac{-3}{8} & \dots & -3 \cdot \frac{1}{2}^{n-2} \\ 0 & 0 & 0 & 0 & 4 & 2 & 1 & \frac{1}{2} & \dots & 2 \cdot \frac{1}{2}^{n-3} \\ \hline 0 & 0 & 4 & -4 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \dots & 2 \left[\frac{1}{2} \right]^{n-2} \dots \left[\frac{1}{2} \right]^{n-3} \left[2 \frac{1}{2}^2 - 3 \frac{1}{2} + 2 \right] \end{bmatrix}$$

IIR convolution



$$y[n] = \left[0 \quad 0 \quad 4 \quad -4 \quad 2 \quad 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \dots 2 \cdot \left[\frac{1}{2} \right]^{n-2} \right]$$

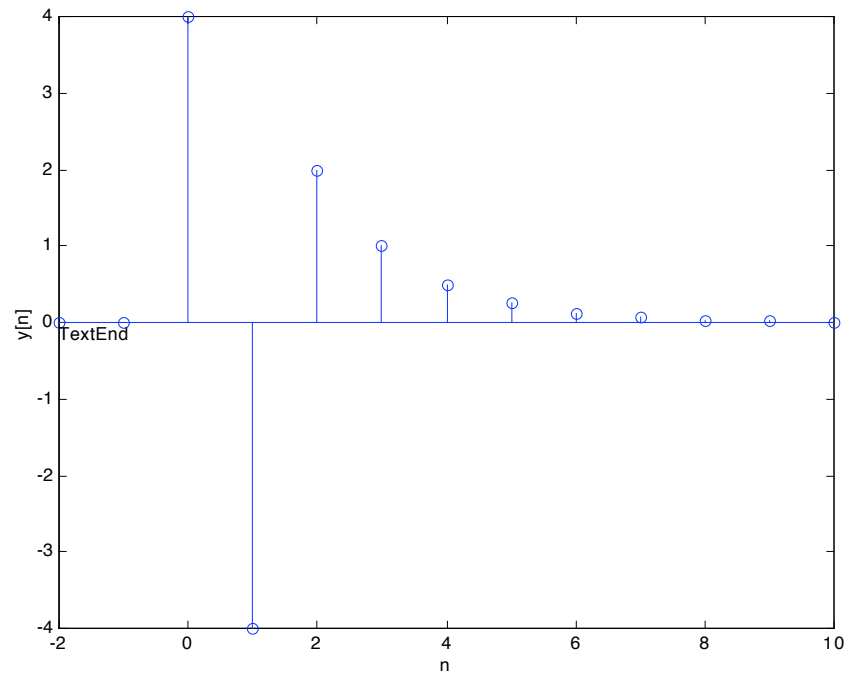
IIR convolution

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$

$$h[n] = \left[0 \quad 0 \quad 2 \quad 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \dots 2 \cdot \frac{1}{2}^n \right]$$

$$y[n] = h[n] * x[n]$$

$$y[n] = \left[0 \quad 0 \quad 4 \quad -4 \quad 2 \quad 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \dots 2 \cdot \left[\frac{1}{2}\right]^{n-2} \right]$$



Frequency response

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

Complex exponential input



$$y[n] = \sum_{k=0}^{\infty} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\hat{\omega}k} \right) Ae^{j\phi}e^{j\hat{\omega}n}$$

$$= \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

frequency response

let

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k]e^{j\hat{\omega}k}$$

This sum must converge for $\mathcal{H}(\hat{\omega})$ to exist.

$\mathcal{H}(\hat{\omega})$ is the Discrete Time Fourier Transform of the impulse function $h[n]$

Frequency response

FIR

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

$$h[k] = \sum_{k=0}^M b_k \delta[n-k]$$

$$\downarrow$$
$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^M h[k] e^{j\hat{\omega}k}$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^M b_k e^{j\hat{\omega}k}$$

Easy to go from difference equation to frequency response because $h[n]$ finite length and $h[n] = [b_0, b_1, \dots]$.

IIR

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

$\downarrow$ X Argh!

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

Tough to go from diff.eqn. to freq. response because $h[n]$ infinite length, $h[n]=f(a_l, b_k)$ is complicated, and $\mathcal{H}(\hat{\omega})$ may be unbounded.

temporal space - n

complex frequency space- z

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

z-transform

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}} = \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$$

Fourier
transform

Argh!
@#! road block

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

frequency space - ω

Hurray!

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

The frequency response is
H(z) evaluated on unit circle

Benefits of z-plane and z-transforms:

1. Get around road block by using z-plane and z-transforms.

Compute system function from diff.eq. coefficients, then evaluate on the unit circle to find the frequency response.

2. z-plane (pole/zeros) will tell us if system stable and frequency response exists.

3. By using z-transforms, solution to diff.eq goes from solving convolution in n-space to solving algebraic equations in z-domain (easier).

And lots more...!

Frequency response

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

Complex exponential input



$$y[n] = \sum_{k=0}^{\infty} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{\infty} h[k]e^{-j\hat{\omega}k} \right) Ae^{j\phi}e^{j\hat{\omega}n}$$

$$= \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

frequency response

let

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k]e^{j\hat{\omega}k}$$

This sum must converge for $\mathcal{H}(\hat{\omega})$ to exist.

$\mathcal{H}(\hat{\omega})$ is the Discrete Time Fourier Transform of the impulse function $h[n]$

System function

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = z^n$$



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{(n-k)}$$

power sequence
of complex numbers

$$= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) z^n$$

let

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$= H(z)z^n$$

$$= |H(z)|e^{j\angle H(z)}z^n$$

scaled and shifted
power sequence

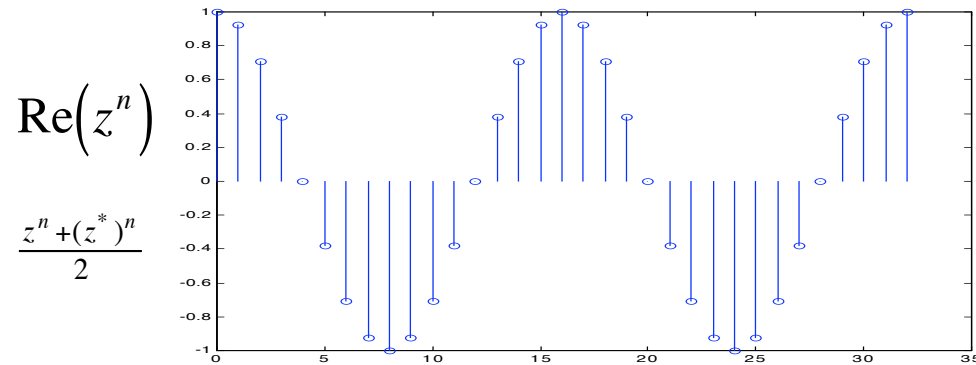
LTI

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

system function

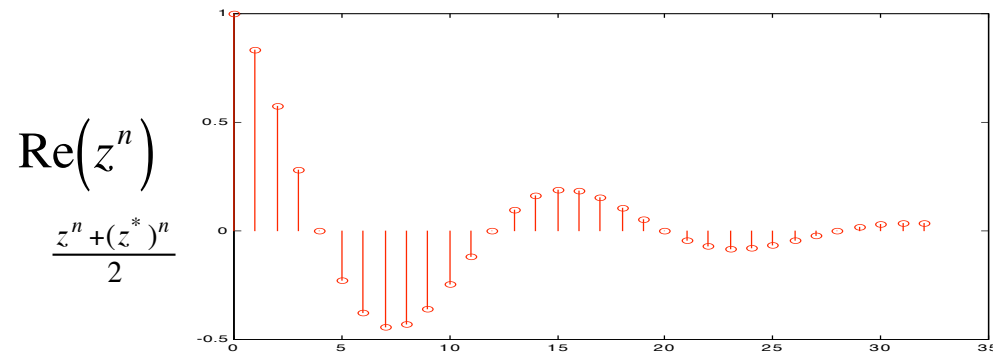
characteristic functions of LTI systems

$$z^n$$



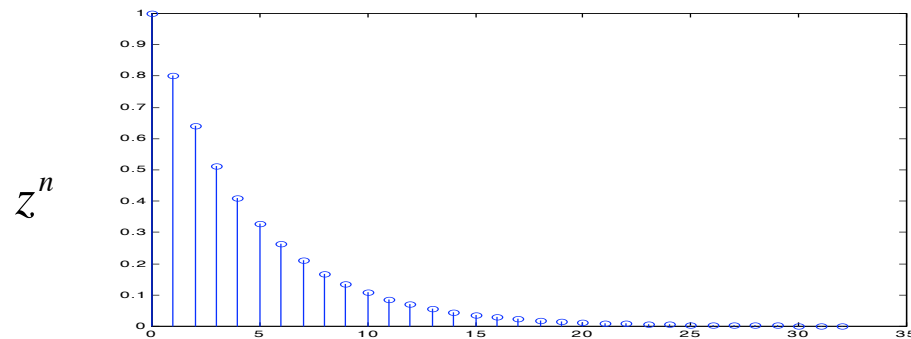
$$z = e^{j\frac{2\pi}{16}}$$

sinusoid



$$z = 0.9e^{j\frac{2\pi}{16}}$$

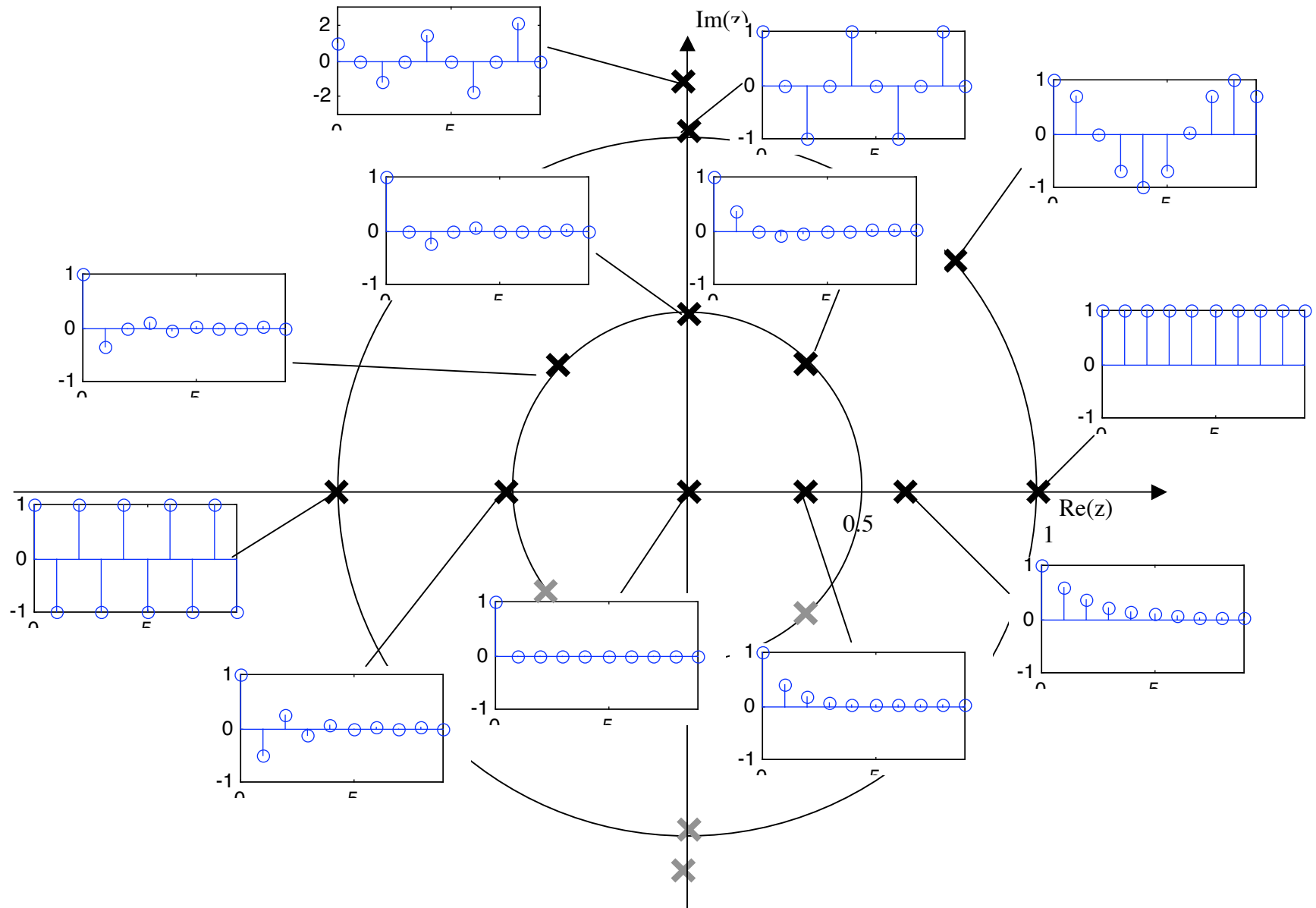
damped sinusoid



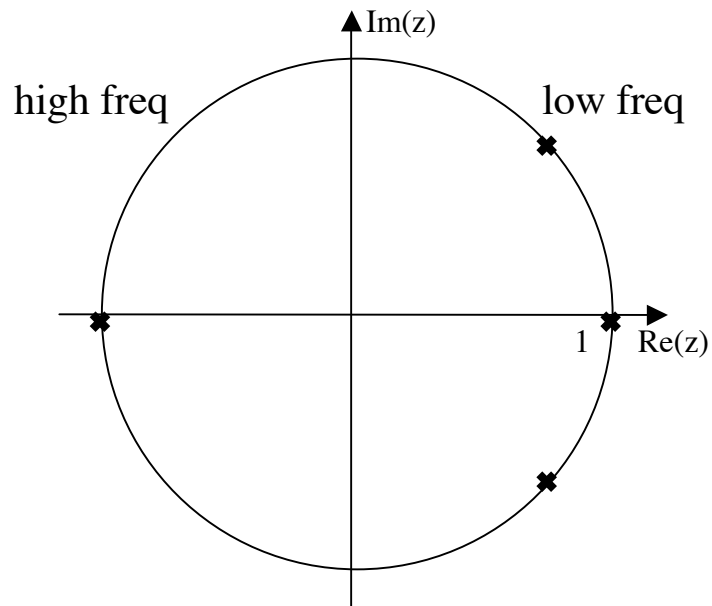
$$z = 0.8$$

exponentials

z-plane and sample responses z^n

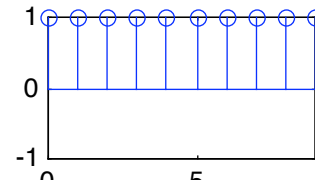


z-plane



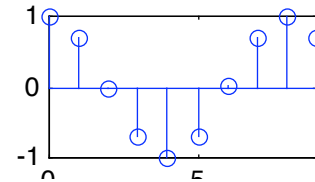
z^n

$$z = 1$$



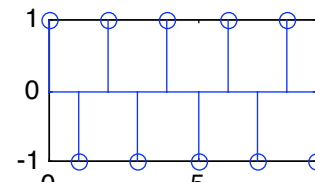
DC, $\omega=0$

$$z = e^{\pm j\hat{\omega}}$$



unit circle
sinusoids

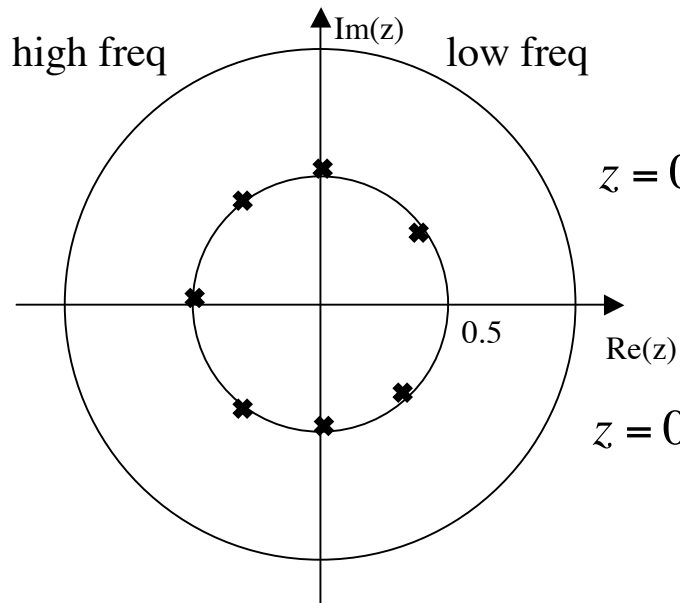
$$z = -1$$



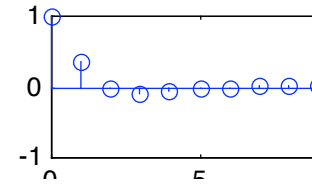
Nyquist sampled
sinusoid

z-plane

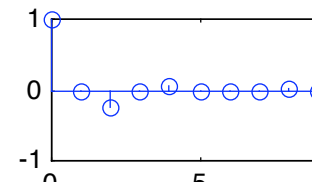
z^n



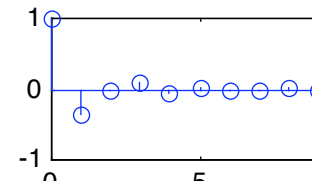
$$z = 0.5e^{\pm j\frac{\pi}{4}}$$



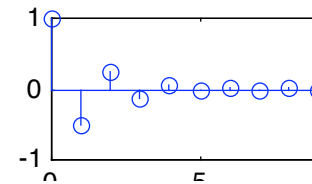
$$z = 0.5e^{\pm j\frac{\pi}{2}}$$



$$z = 0.5e^{\pm j\frac{3\pi}{4}}$$



$$z = -0.5$$

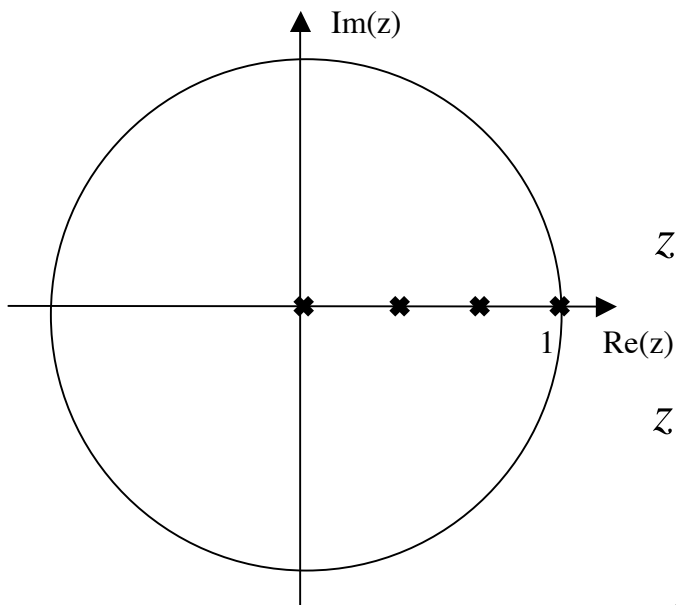


$$|z| < 1$$

Damped sinusoids

Nyquist sampled
damped sinusoid

z-plane



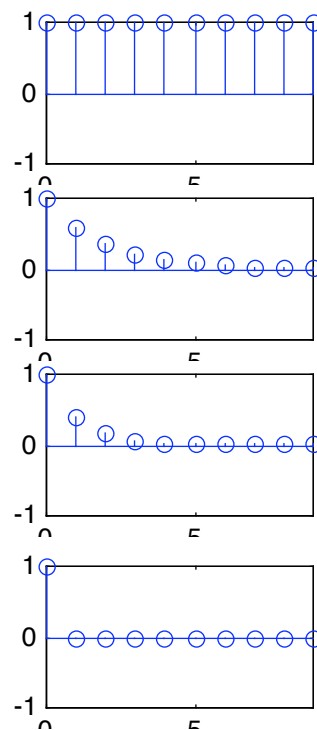
$$z = 1$$

$$z = 0.6$$

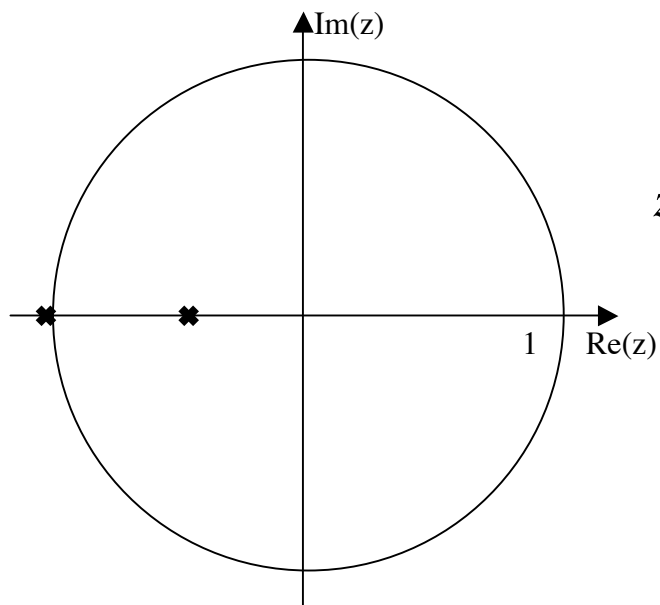
$$z = 0.4$$

$$z = 0$$

z^n

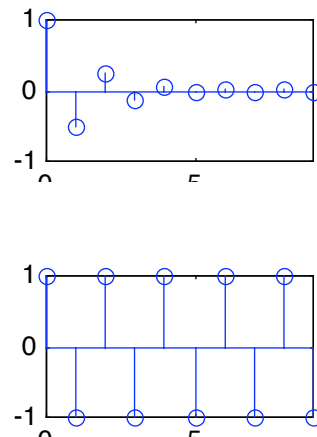


+Re(z) axis
exponential
decay



$$z = -0.5$$

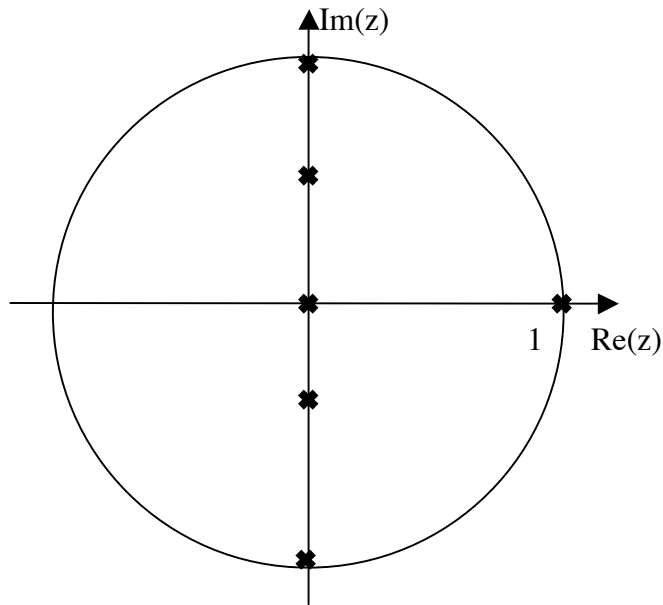
$$z = -1$$



-Re(z) axis
 $\omega_s/2$
2 samples/cycle

Nyquist sampled
signals

z-plane

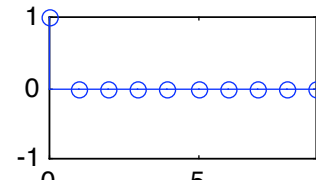
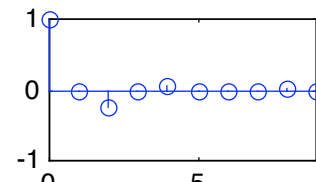
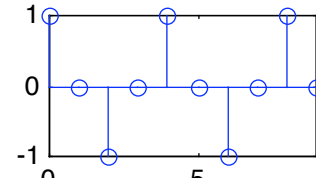


$$z = \pm 1j$$

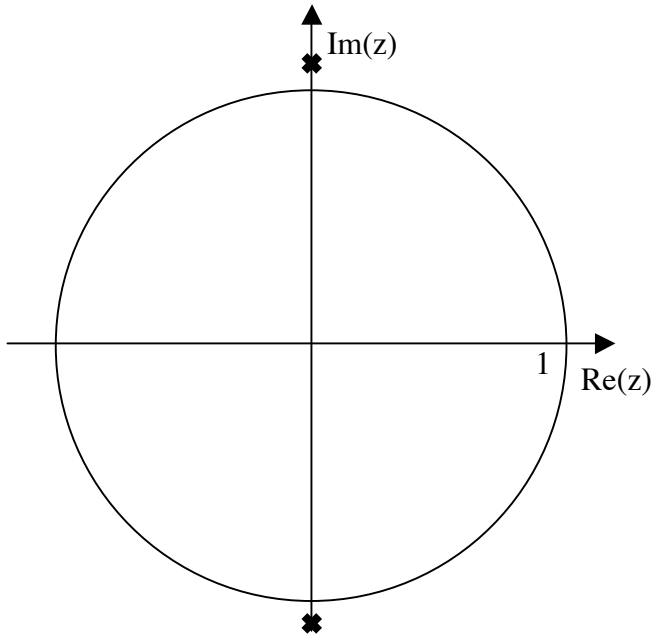
$$z = \pm j0.5$$

$$z = 0$$

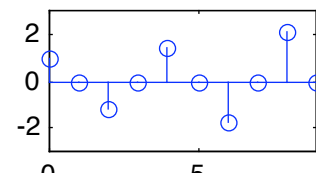
z^n



$\text{Im}(z)$ axis
every other
sample 0



$$z = \pm j1.1$$



$|z| > 1$ unstable

$|z| < 1$ stable

System function

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

convolution

$$x[n] = z^n$$



power sequence
of complex numbers

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right) z^n$$

let

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

$$= H(z)z^n$$

$$= |H(z)|e^{j\angle H(z)}z^n$$

scaled and shifted
power sequence

LTI

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

system function

z-transform

$$x[n] \overset{z}{\Leftrightarrow} X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} \quad \text{definition}$$

Ex.

impulse

$$x[n] = \delta[n] \quad \overset{z}{\Leftrightarrow} \quad X(z) = \sum_{k=-\infty}^{\infty} \delta[k]z^{-k}$$

$$= z^{-0}$$

$$= 1$$

Ex.

$$y[n] = h[n] * x[n]$$

$$= h[n] * \delta[n]$$

$$\overset{z}{\Updownarrow} \quad Y(z) = H(z) \cdot 1$$

$$= H(z)$$

$$\overset{z}{\Updownarrow} \quad y[n] = h[n]$$

$H(z)$ is the z-transform
of the impulse response
 $h[n]$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

LTI

Ex.

n_0 sample delayed impulse

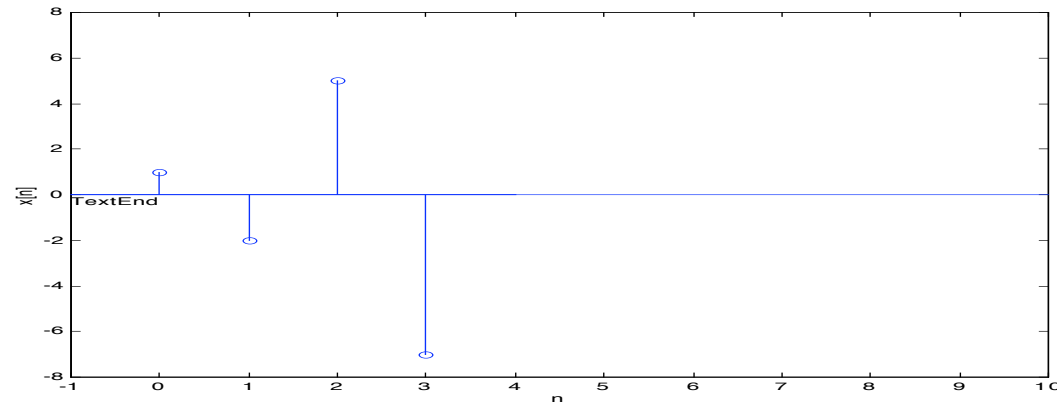
$$x[n] = \delta[n - n_0] \quad \Leftrightarrow$$

n_0^{th} power of z^{-1}

$$X(z) = \sum_{k=-\infty}^{\infty} \delta[k - n_0] z^{-k}$$

$$= z^{-n_0} = \left(z^{-1}\right)^{n_0}$$

Ex. finite response



$$x[n] = 1\delta[n] - 2\delta[n - 1] + 5\delta[n - 2] - 7\delta[n - 3]$$

$$\Uparrow_z$$

$$X(z) = 1 - 2z^{-1} + 5z^{-2} - 7z^{-3}$$

Ex. $y[n] = \frac{1}{3}x[n] + \frac{1}{3}x[n-1] + \frac{1}{3}x[n-2]$ FIR

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

$$\begin{aligned} H(z) &= \sum_{k=0}^2 h[k]z^{-k} \\ &= h[0]z^0 + h[1]z^{-1} + h[2]z^{-2} \\ &= \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} \\ &= \frac{1}{3}\left(z^{-1}\right)^0 + \frac{1}{3}\left(z^{-1}\right)^1 + \frac{1}{3}\left(z^{-1}\right)^2 \quad \text{polynomial in } z^{-1} \end{aligned}$$

$$h[n] = \sum_{k=0}^2 b_k \delta[n-k] \Leftrightarrow H(z) = \sum_{k=0}^2 b_k z^{-k} \quad \text{FIR only!}$$

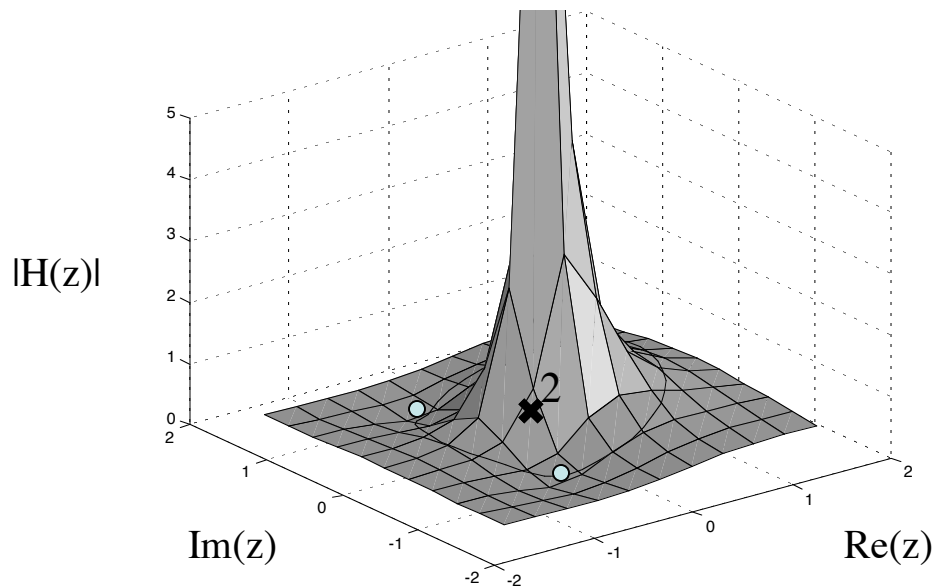
sequence	$\Leftrightarrow$	polynomial	
n-domain (sample space)		z-domain (complex freq space)	

Ex.
$$H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

$$y[n] = H(z)z^n$$

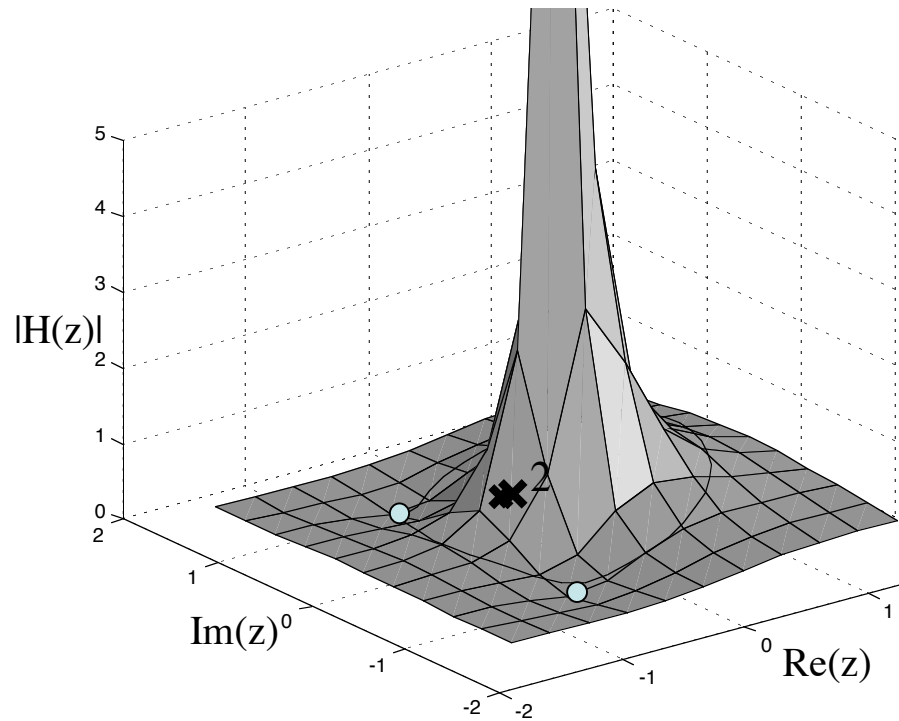
num=0 $H(z) = 0$ $y[n] = 0$ $z^2 + z + 1 = 0$ **zeros**
 $z = \frac{1}{2}(-1 \pm j\sqrt{3}) = e^{\pm j2\pi/3}$ roots of numerator

denom=0 $H(z) = \infty$ $y[n] = \infty$ $z^2 = 0$ **poles**
 $z = 0, 0$ roots of denominator

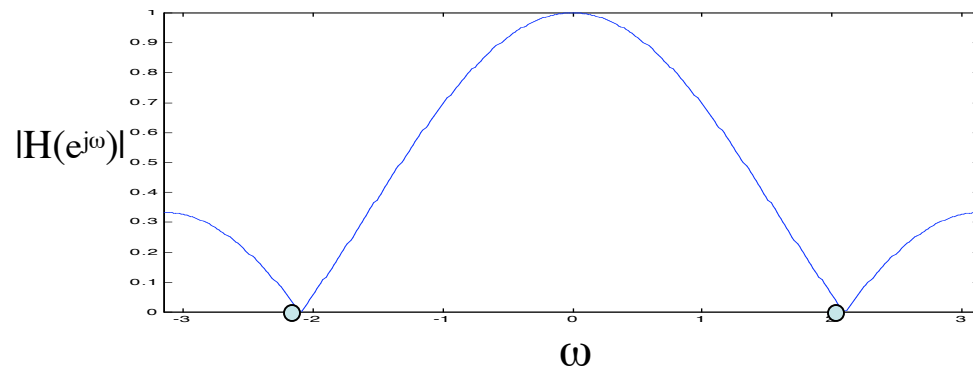
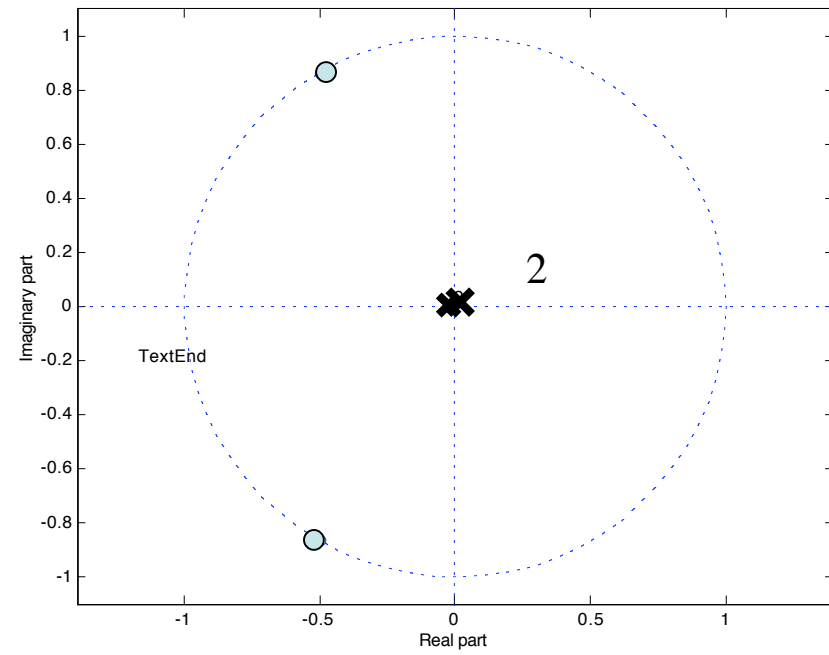


FIR filters only have zeros on unit circle, and poles are either at 0 or ∞ .
 #poles=#zeros
 “extra” zero/poles are at $z=\infty$.

system response $|H(z)|$



pz plot



frequency response

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

The frequency response is
 $H(z)$ evaluated on unit circle

IIR

Ex.

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

$$h[n] = \sum_{l=1}^N a_l h[n-l] + \sum_{k=0}^M b_k \delta[n-k]$$



$$H(z) = \sum_{l=1}^N a_l H(z) z^{-l} + \sum_{k=0}^M b_k z^{-k}$$

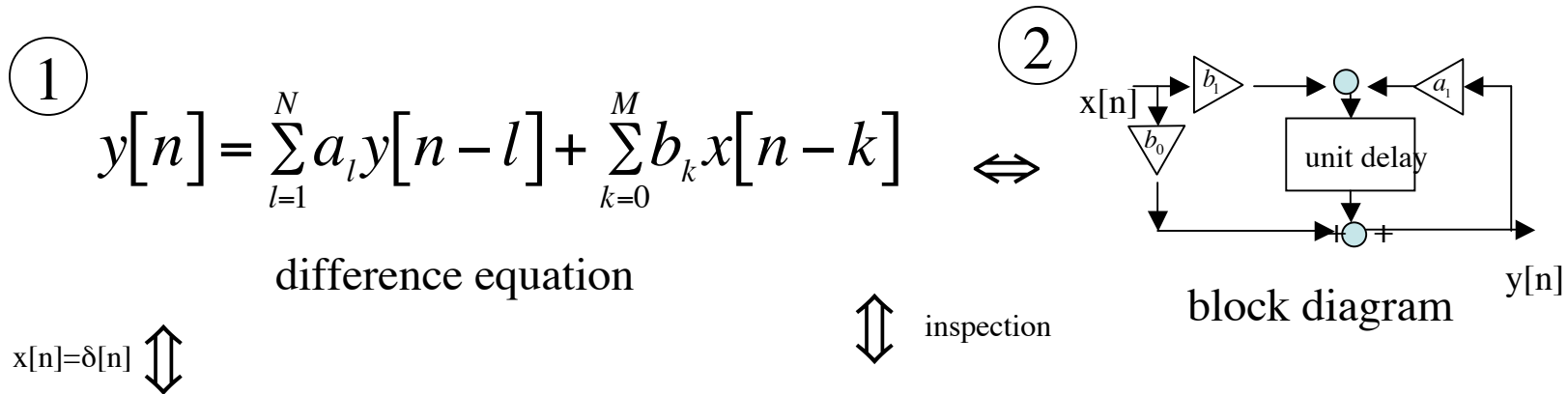
$$H(z) \left(1 - \sum_{l=1}^N a_l z^{-l} \right) = \sum_{k=0}^M b_k z^{-k}$$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}}$$

freqz(b,a)

sequence	↔	polynomial	
n-domain (sample space)		z-domain (complex freq space)	

Equivalent ways to represent the system



③
$$h[n] = y[n] \Big|_{x[n] = \delta[n]}$$

impulse response
sequence

$\xLeftrightarrow{z} H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}} = \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$

④ system function
polynomial

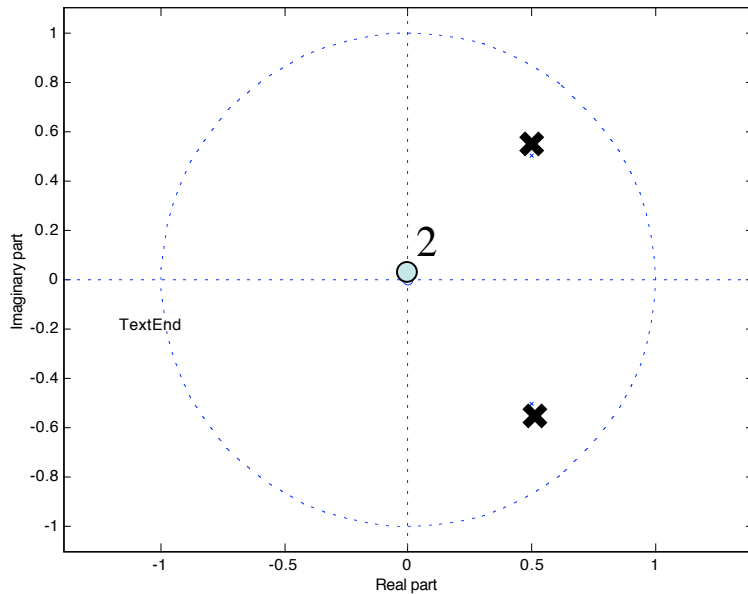
$z = e^{j\omega}$ $\Updownarrow$

pole-zero locations ⑤

⑥
$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

frequency response

All poles must be inside unit circle for $\mathcal{H}(\omega)$ to converge and the system to be stable.
(FIR filter always stable)



difference equation IIR filter

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}}$$

$$H(z) = \frac{1}{1 - z^{-1} + 0.5z^{-2}} = \frac{z^2}{z^2 - z + 0.5}$$

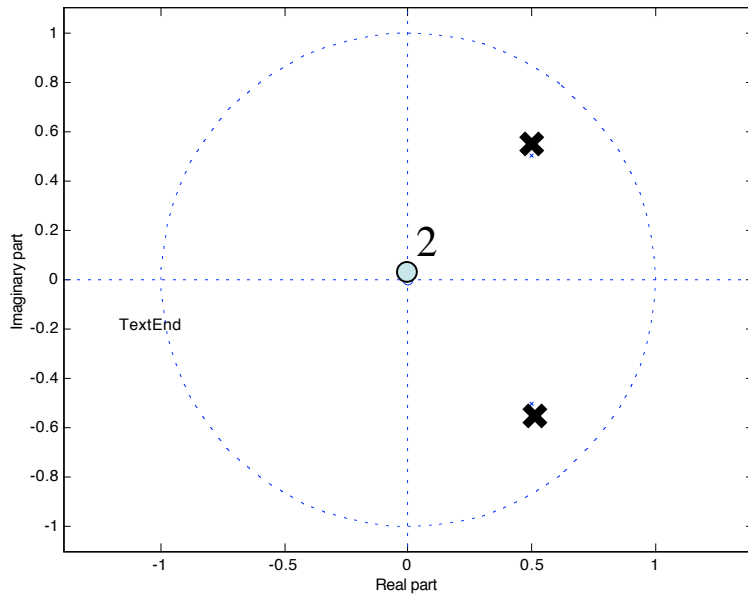
$$= \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$$

$$= \frac{z \cdot z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

frequency response
 $\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

zeros: 0, 0

poles: $0.5 \pm j0.5$



zeros: 0, 0

IIR filter

poles: $0.5 \pm j0.5$

$$\begin{aligned}
 H(z) &= \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})} \\
 &= \frac{z \cdot z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)} \\
 &= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}
 \end{aligned}$$

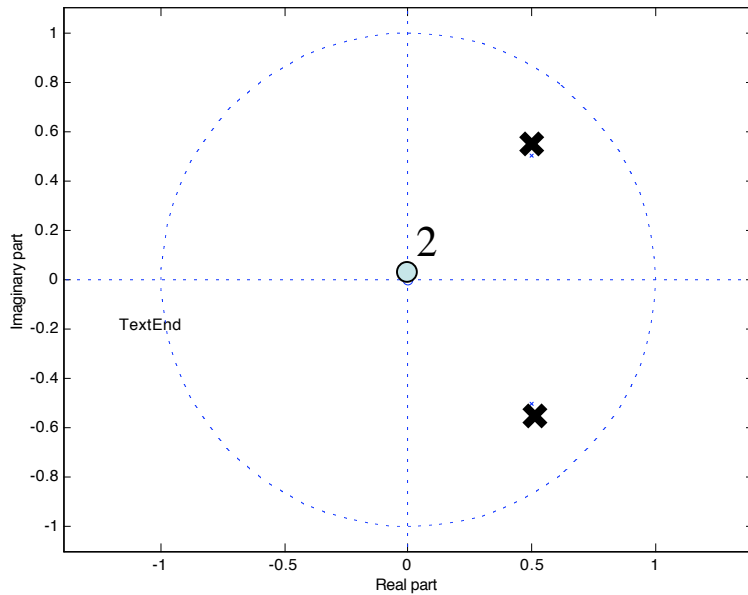
$$\begin{aligned}
 \frac{1}{1 - z^{-1} + 0.5z^{-2}} &= \frac{1}{1 - z^{-1} + 0.5z^{-2}} \cdot \frac{z^2}{z^2} \\
 &= \frac{z^2}{z^2 - z + 0.5}
 \end{aligned}$$

frequency response
 $\mathcal{H}(\omega) = H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}}$$

difference equation

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$



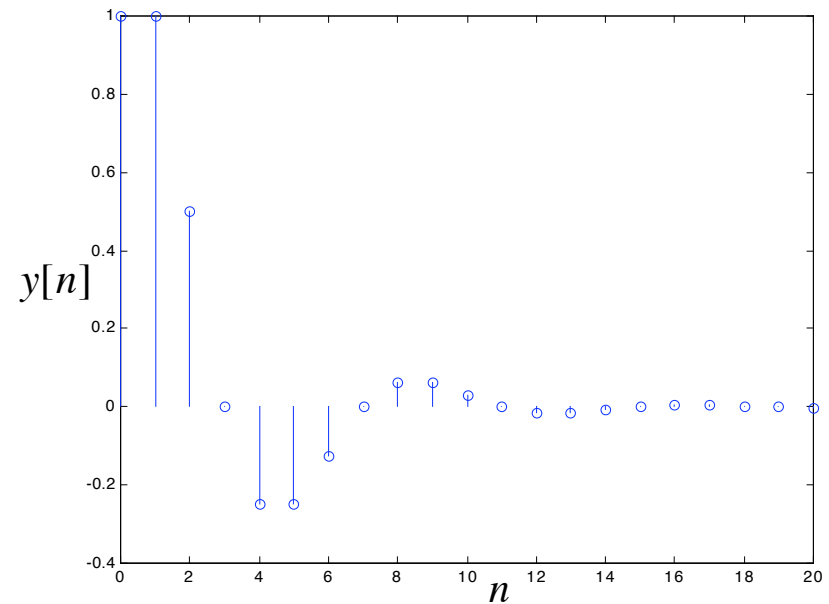
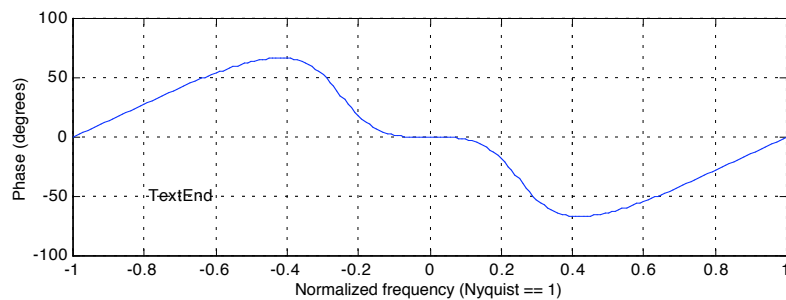
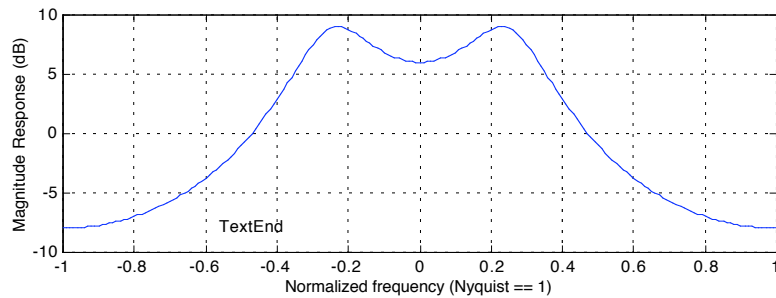
zeros: 0, 0 IIR filter
poles: $0.5 \pm j0.5$

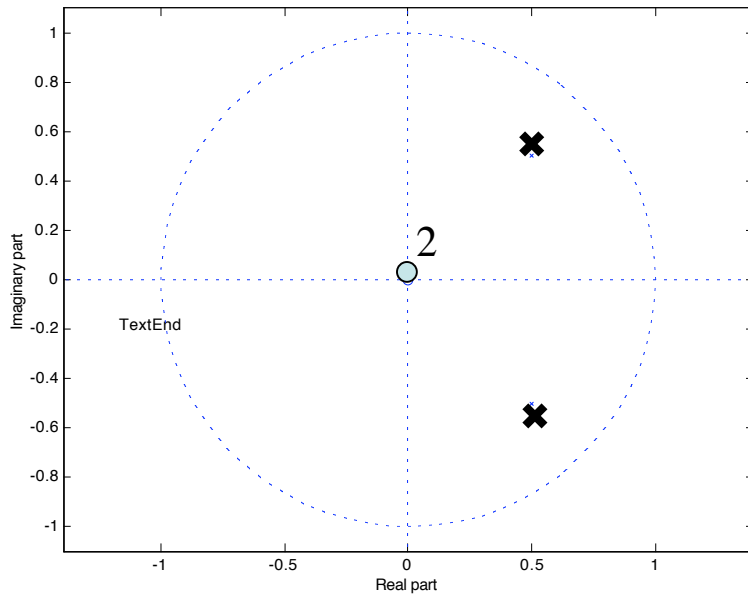
$$H(z) = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

$$H(\hat{\omega}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$





zeros: 0, 0 IIR filter
poles: $0.5 \pm j0.5$

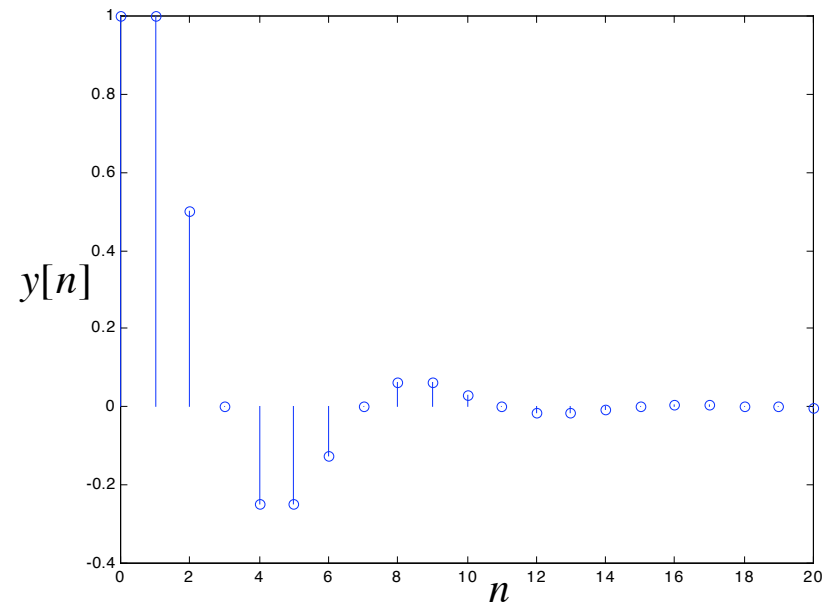
$$H(z) = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

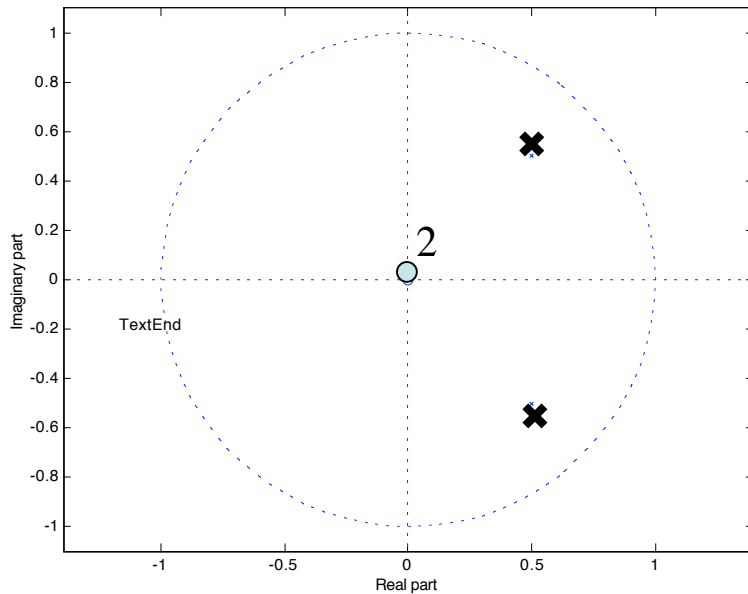
$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

$$H(\hat{\omega}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$

To go from p-z map to
impulse response:
Look at locations of
poles. Closer poles are
to zeros, the weaker
their response.





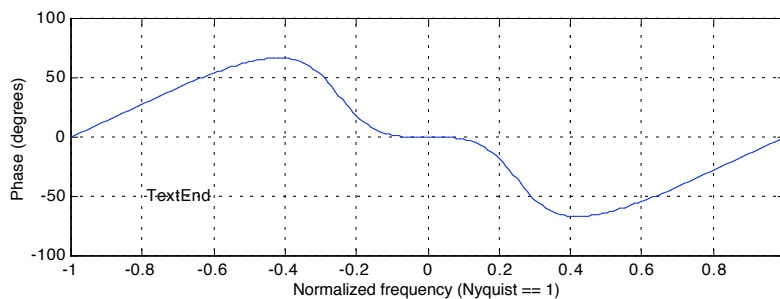
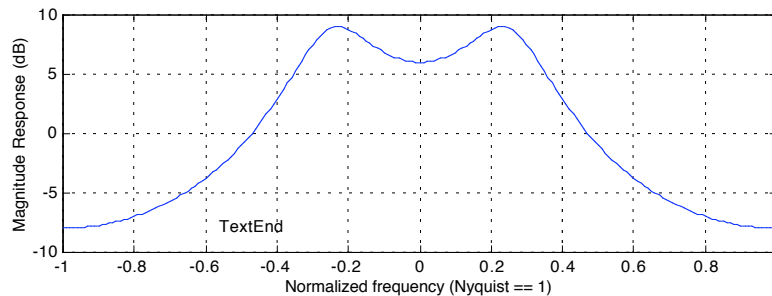
zeros: 0, 0 IIR filter
poles: $0.5 \pm j0.5$

$$H(z) = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

$$= \frac{z^2}{z^2 - z + 0.5} = \frac{1}{1 - z^{-1} + 0.5z^{-2}}$$

$$H(\hat{\omega}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

$$y[n] = x[n] + y[n-1] - 0.5y[n-2]$$



From zp map to freq response:
Go around unit circle from DC to Nyquist. Response goes down near zeros, goes up near poles.
Closer poles are to zeros, the weaker their effect.

z-transform

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

sequence	$\Leftrightarrow$	polynomial
n-domain (sample space)		z-domain (complex freq space)

	$h[n] \Leftrightarrow H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}}$	
impulse response sequence		system function polynomial

$H(z)$ is the z-transform of the impulse response $h[n]$.

LTI

z-transform

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

sequence	$\Leftrightarrow$	polynomial
n-domain (sample space)		z-domain (complex freq space)

Why're we interested?

$$y[n] = \sum_{k=0}^M h[k] x[n-k] \quad \text{convolution sum}$$

$$Y(z) = H(z)X(z) \quad \text{polynomial multiplication}$$

z-transform

$$x[n] \Leftrightarrow X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

impulse

$$x[n] = \delta[n] \quad \Leftrightarrow \quad X(z) = \sum_{k=-\infty}^{\infty} \delta[k]z^{-k}$$
$$= z^{-0}$$
$$= 1$$

Ex.

$$y[n] = h[n] * x[n]$$
$$= h[n] * \delta[n]$$

$$\begin{array}{c} \Downarrow \quad z \\ Y(z) = H(z) \cdot 1 \\ = H(z) \end{array}$$

$$\begin{array}{c} \Downarrow \quad z \\ y[n] = h[n] \end{array}$$

$H(z)$ is the z-transform
of the impulse response
 $h[n]$

LTI

n_0 sample delayed impulse

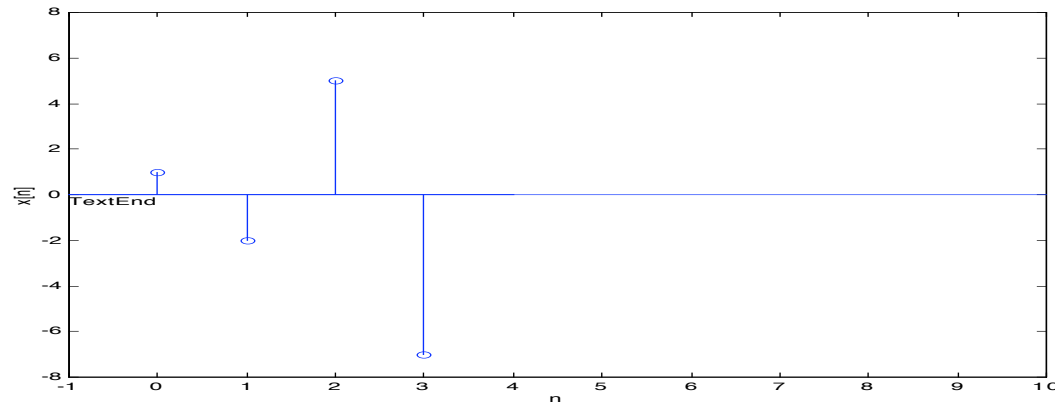
$$x[n] = \delta[n - n_0] \quad \Leftrightarrow$$

n_0^{th} power of z^{-1}

$$X(z) = \sum_{k=-\infty}^{\infty} \delta[k - n_0] z^{-k}$$

$$= z^{-n_0} = \left(z^{-1}\right)^{n_0}$$

Ex. finite response



$$x[n] = 1\delta[n] - 2\delta[n - 1] + 5\delta[n - 2] - 7\delta[n - 3]$$

$$\Uparrow_z$$

$$X(z) = 1 - 2z^{-1} + 5z^{-2} - 7z^{-3}$$

Infinite signals

$$x[n]u[n] \Leftrightarrow X(z) = \sum_{k=0}^{\infty} x[k]z^{-k} \quad \text{right-sided signals}$$

$$\begin{aligned} x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) &= \sum_{k=0}^{\infty} a^k z^{-k} \\ &= \sum_{k=0}^{\infty} (az^{-1})^k \\ &= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 \dots \\ &\quad \text{geometric series} \end{aligned}$$

Infinite signals

$$x[n] = a^n u[n] \quad \Leftrightarrow \quad X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$
$$= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 \dots$$

geometric series

$$X(z) = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 \dots$$
$$- az^{-1}X(z) = -az^{-1} - (az^{-1})^2 - (az^{-1})^3 - (az^{-1})^4 \dots$$

$$(1 - az^{-1})X(z) = 1$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad \begin{array}{l} |az^{-1}| < 1 \\ \text{or } |z| > |a| \end{array} \quad \text{region of convergence}$$

Infinite series:

IIR filter

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

$x[n]=0$ $n<0$
right sided

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$

region of convergence

Finite series:

FIR filter

$$x[n] = a^n (u[n - M] - u[n - N]) \Leftrightarrow X(z) = \sum_{k=M}^{N-1} a^k z^{-k}$$

$$X(z) = \frac{(az^{-1})^M - (az^{-1})^N}{1 - az^{-1}}$$

all z region of convergence

$$\lim_{z \rightarrow a} X(z) = N - M$$

Ex. $x[n] = 1u[n]$ $X(z) = ?$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k}$$

$$= \sum_{k=0}^{\infty} (z^{-1})^k \quad \text{let } a = 1$$

$$= \frac{1}{1 - z^{-1}} \quad |z| > |1|$$

We know:

$$\sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$

roc

Ex. $x[n] = \cos(\hat{\omega}n)u[n]$ $X(z) = ?$

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} = \sum_{k=0}^{\infty} \cos(\hat{\omega}k)z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{j\hat{\omega}k} + e^{-j\hat{\omega}k}}{2} \right) z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} e^{j\hat{\omega}k} z^{-k} + \frac{1}{2} \sum_{k=0}^{\infty} e^{-j\hat{\omega}k} z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(e^{j\hat{\omega}} z^{-1} \right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(e^{-j\hat{\omega}} z^{-1} \right)^k$$

let $a = e^{j\hat{\omega}}$

let $a = e^{-j\hat{\omega}}$

$$= \frac{1}{2} \frac{1}{1 - e^{j\hat{\omega}} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\hat{\omega}} z^{-1}}$$

We know:

$$\sum_{k=0}^{\infty} \left(a z^{-1} \right)^k = \frac{1}{1 - a z^{-1}}$$

$$|z| > |a|$$

roc

intersection of roc's

$$|z| > |e^{j\hat{\omega}}| \cap |e^{-j\hat{\omega}}|$$

$$|z| > |1|$$

Ex. $x[n] = \cos(\hat{\omega}n)u[n]$ $X(z) = ?$ Cont.

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} = \sum_{k=0}^{\infty} \cos(\hat{\omega}k)z^{-k}$$

$$= \frac{1}{2} \frac{1}{1 - e^{j\hat{\omega}} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-j\hat{\omega}} z^{-1}}$$

$$= \frac{1}{2} \frac{z}{z - e^{j\hat{\omega}}} + \frac{1}{2} \frac{z}{z - e^{-j\hat{\omega}}}$$

$$= \frac{1}{2} z \frac{2z - e^{-j\hat{\omega}} - e^{j\hat{\omega}}}{z^2 - (e^{-j\hat{\omega}} + e^{j\hat{\omega}})z + 1}$$

$$= z \frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1}$$

$$|z| > 1$$

Table of z-transforms

signal	z-transform	ROC
$\delta[n - k]$	z^{-k}	<i>all</i> z
$u[n]$	$\frac{z}{z - 1}$	$ z > 1$
$n^2 u[n]$	$\frac{z(z + 1)}{(z - 1)^3}$	$ z > 1$
$\gamma^n \cos(\alpha n) u[n]$	$\frac{z(z - \gamma \cos \alpha)}{z^2 - (2\gamma \cos \alpha)z + \gamma^2}$	$ z > \gamma $
$\gamma^n \sin(\alpha n) u[n]$	$\frac{z\gamma \sin \alpha}{z^2 - (2\gamma \cos \alpha)z + \gamma^2}$	$ z > \gamma $

z-transform properties

$$a_1 x_1[n] + a_2 x_2[n] \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

linearity/superposition

$$x[n - m] \Leftrightarrow z^{-m} X(z)$$

sample shift

$$h[n] * x[n] \Leftrightarrow H(z) X(z)$$

sample domain convolution
z domain multiplication

$$nx[n] \Leftrightarrow -z \frac{dX(z)}{dz}$$

multiply by a ramp
z domain differentiation

$$a^n x[n] \Leftrightarrow X\left(\frac{z}{a}\right)$$

multiply by an exponential
z domain scaling

Ex. $x[n] = na^n u[n]$ $X(z) = ?$

We already know

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}} \quad \text{and} \quad ny[n] \Leftrightarrow -z \frac{dY(z)}{dz}$$

$$\begin{aligned} \frac{d}{dz} \left(\frac{1}{1 - az^{-1}} \right) &= \frac{d}{dz} \left(\frac{z}{z - a} \right) \\ &= \frac{-a}{(z - a)^2} = \frac{-az^{-2}}{(1 - az^{-1})^2} \end{aligned}$$

$$\begin{aligned} na^n u[n] &\Leftrightarrow -z \frac{az^{-2}}{(1 - az^{-1})^2} \\ &\Leftrightarrow -\frac{az^{-1}}{(1 - az^{-1})^2} \end{aligned}$$

Next:

Solve difference equation using z-transforms.

Convolution problem in the temporal domain becomes an algebra problem in the z-domain.

Need to know how to convert from z-domain back to temporal domain (inverse z-transform).

$$\begin{aligned}
y[n] &= \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n-2] \\
h[n] &= \frac{1}{3} \delta[n] + \frac{1}{3} \delta[n-1] + \frac{1}{3} \delta[n-2] \\
x[n] &= \cos(\hat{\omega}n)u[n]
\end{aligned}$$

$$y[n] = h[n] * x[n] \Leftrightarrow Y(z) = H(z)X(z)$$

$$H(z) = \frac{1}{3} + \frac{1}{3} z^{-1} + \frac{1}{3} z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

$$X(z) = z \frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1}$$

$$Y(z) = \frac{z^2 + z + 1}{3z^2} \left(z \frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1} \right)$$

$$y[n] = \mathcal{Z}^{-1} \left[\frac{z^2 + z + 1}{3z^2} \left(z \frac{z - \cos(\hat{\omega})}{z^2 - 2\cos(\hat{\omega})z + 1} \right) \right]$$