

Frequency response

FIR

$$y[n] = \sum_{k=0}^M b_k x[n-k]$$

$$h[k] = \sum_{k=0}^M b_k \delta[n-k]$$

$$\downarrow$$
$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^M h[k] e^{j\hat{\omega}k}$$

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^M b_k e^{j\hat{\omega}k}$$

Easy to go from difference equation to frequency response because $h[n]$ finite length and $h[n] = [b_0, b_1, \dots]$.

IIR

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

$\downarrow$ ~~X~~ Argh!

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

Tough to go from diff.eqn. to freq. response because $h[n]$ infinite length, $h[n]=f(a_l, b_k)$ is complicated, and $\mathcal{H}(\hat{\omega})$ may be unbounded.

temporal space - n

complex frequency space- z

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

$$h[k] \neq \sum_{k=0}^{\infty} b_k \delta[n-k]$$

z-transform

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}} = \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$$

Fourier
transform

↓ Argh!
@#! road block

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$$

frequency space - ω

← Hurray!

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

The frequency response is
H(z) evaluated on unit circle

Benefits of z-plane and z-transforms:

1. Get around road block by using z-plane and z-transforms.

Compute system function from diff.eq. coefficients, then evaluate on the unit circle to find the frequency response.

2. z-plane (pole/zeros) will tell us if system stable and frequency response exists.

3. By using z-transforms, solution to diff.eq goes from solving convolution in n-space to solving algebraic equations in z-domain (easier).

And lots more...!

Infinite signals

$$\begin{aligned} x[n] = a^n u[n] & \Leftrightarrow X(z) = \sum_{k=0}^{\infty} a^k z^{-k} \\ \text{x[n]=0 n<0} & \\ \text{right sided} & \\ & = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 \dots \\ & \text{geometric series} \end{aligned}$$

$$\begin{aligned} X(z) &= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 \dots \\ - az^{-1} X(z) &= - az^{-1} - (az^{-1})^2 - (az^{-1})^3 - (az^{-1})^4 \dots \end{aligned}$$

$$(1 - az^{-1})X(z) = 1$$

$$\begin{aligned} X(z) &= \frac{1}{1 - az^{-1}} & |az^{-1}| < 1 & \text{region of convergence} \\ \text{or } |z| &> |a| \end{aligned}$$

Infinite signals

$$x[n] = -a^n u[-n-1] \Leftrightarrow X(z) = - \sum_{k=-\infty}^{-1} a^k z^{-k}$$

$x[n] = 0 \quad n \geq 0$
left sided

$$= -\frac{1}{a} z - \left(\frac{1}{a} z\right)^2 - \left(\frac{1}{a} z\right)^3 \dots$$

geometric series

$$X(z) = -\frac{1}{a} z - \left(\frac{1}{a} z\right)^2 - \left(\frac{1}{a} z\right)^3 \dots$$
$$- a z^{-1} X(z) = 1 + \frac{1}{a} z + \left(\frac{1}{a} z\right)^2 + \left(\frac{1}{a} z\right)^3 + \left(\frac{1}{a} z\right)^4 \dots$$

$$(1 - a z^{-1}) X(z) = 1$$

$$X(z) = \frac{1}{1 - a z^{-1}} \quad \left| \frac{1}{a} z \right| < 1 \quad \text{region of convergence}$$

or $|z| < |a|$

Infinite series:

$$x[n] = a^n u[n] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad \text{region of convergence}$$

$x[n]=0 \ n<0$
right sided

$$x[n] = -a^n u[-n - 1] \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}} \quad |z| < |a|$$

$x[n] = 0 \ n \geq 0$
left sided

Finite series:

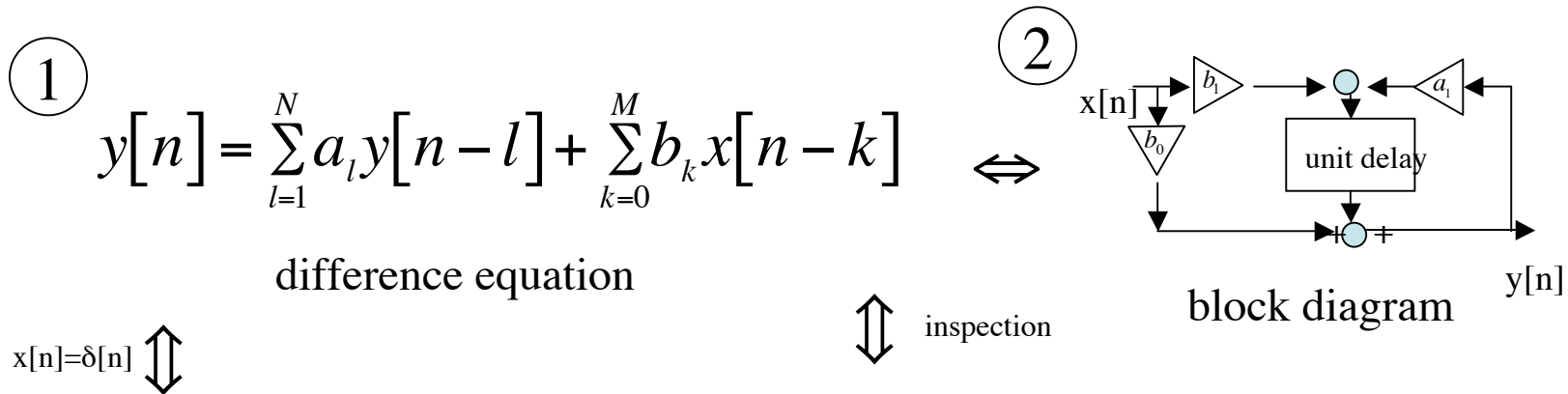
$$x[n] = a^n (u[n - M] - u[n - N]) \Leftrightarrow X(z) = \sum_{k=M}^{N-1} a^k z^{-k}$$

$$X(z) = \frac{(az^{-1})^M - (az^{-1})^N}{1 - az^{-1}}$$

all z region of convergence

$$\lim_{z \rightarrow a} X(z) = N - M$$

Equivalent ways to represent the system



③
$$h[n] = y[n] \Big|_{x[n] = \delta[n]}$$

impulse response
sequence

$\xleftrightarrow{z} H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}} = \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$

④ system function
polynomial

$z = e^{j\omega}$ ↔

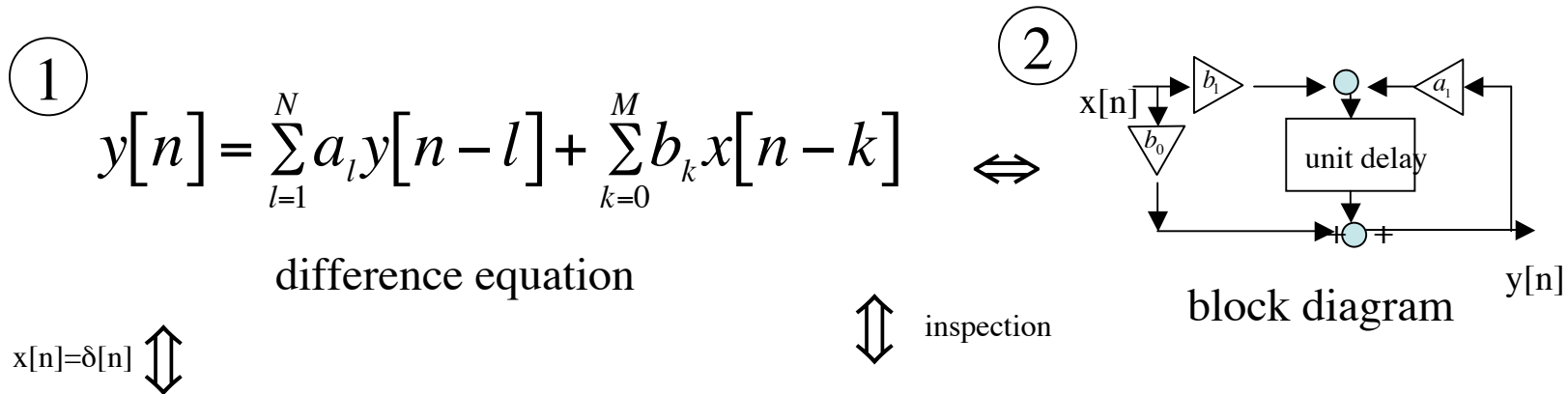
pole-zero
locations ⑤

⑥
$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

frequency response

All poles must be inside
unit circle for $\mathcal{H}(\omega)$
to converge and the system
to be stable. (causal system)
(FIR filter always stable)

Equivalent ways to represent the system



③
$$h[n] = y[n] \Big|_{x[n] = \delta[n]} \xLeftrightarrow{z} H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}} = \frac{\prod_{i=0}^M (z - z_{zi})}{\prod_{i=0}^N (z - z_{pi})}$$

impulse response sequence

④ system function polynomial $\Updownarrow z = e^{j\omega}$ ⑤ pole-zero locations

⑥
$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

frequency response

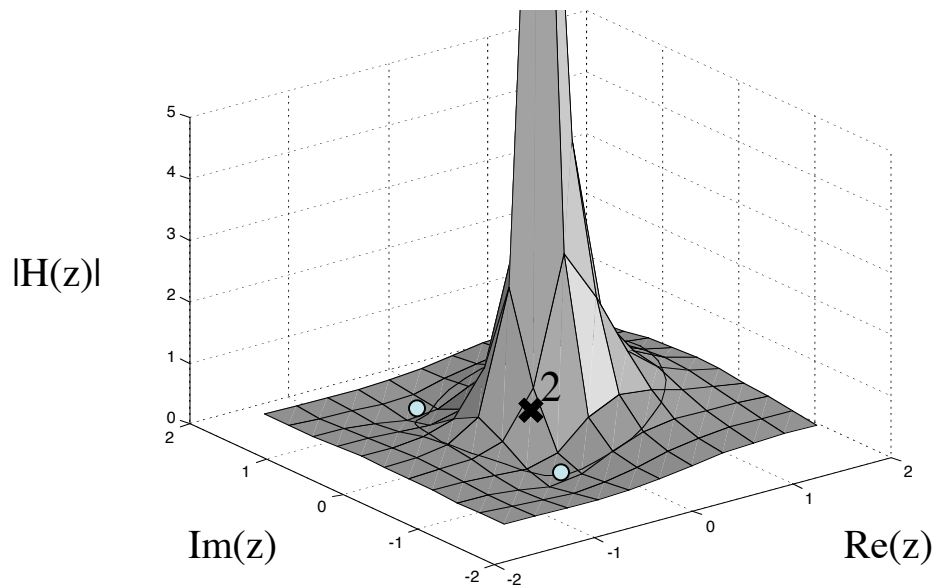
The region of convergence must contain the unit circle for $\mathcal{H}(\omega)$ to converge and the system to be stable. (general)
(FIR filter always stable)

Ex.
$$H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

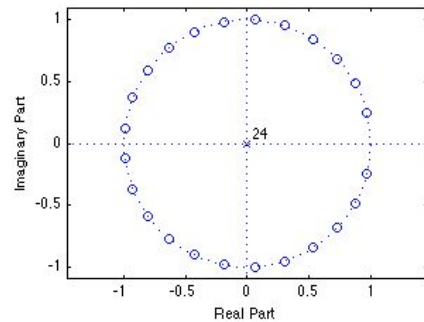
$$y[n] = H(z)z^n$$

num=0 $H(z) = 0$ $y[n] = 0$ $z^2 + z + 1 = 0$ **zeros**
 $z = \frac{1}{2}(-1 \pm j\sqrt{3}) = e^{\pm j2\pi/3}$ roots of numerator

denom=0 $H(z) = \infty$ $y[n] = \infty$ $z^2 = 0$ **poles**
 $z = 0, 0$ roots of denominator



*FIR L point summer/averager
 only has zeros on unit circle
~~FIR filters only have zeros~~
~~on unit circle, and poles~~
 are either at 0 or ∞ .
 #poles=#zeros
 “extra” zero/poles are at $z=\infty$.

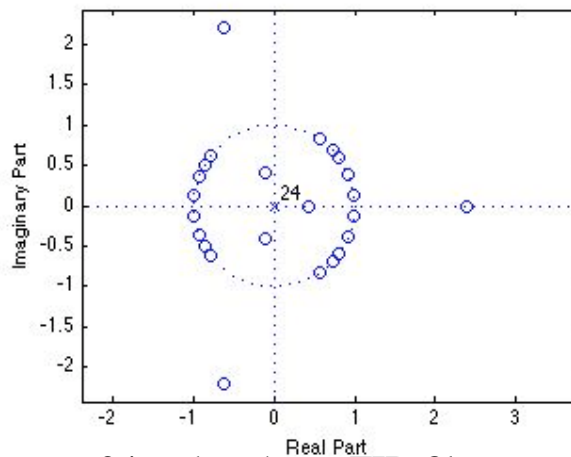
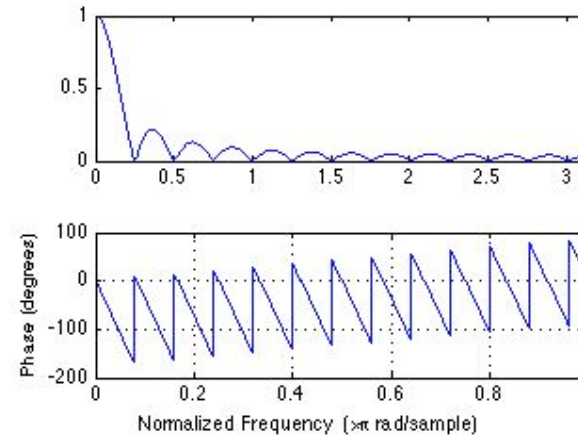


25-pt averager lowpass FIR filter

*poles all at zero (or ∞)

*zeros evenly distributed on unit circle

*missing zero at DC (lowpass)



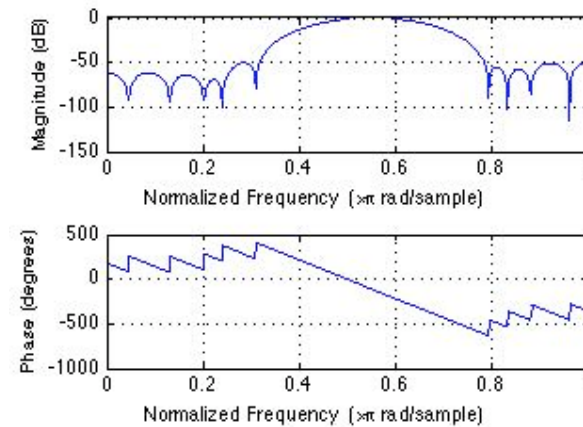
24-pt bandpass FIR filter

*poles all at zero (or ∞)

*zeros not necessarily on unit circle

* Only pole locations affect stability

`b=fir1(24, [.45 .65], 'bandpass');`

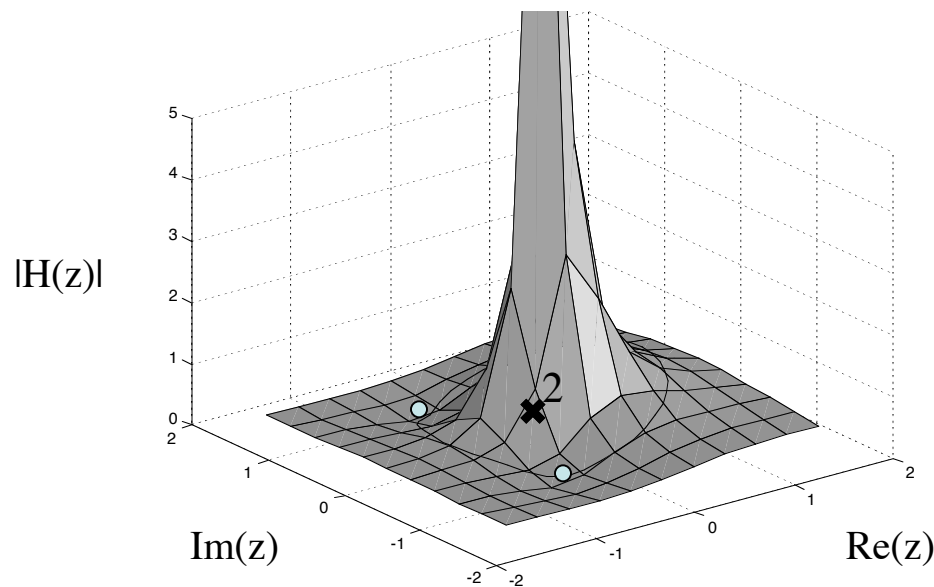


Ex.
$$H(z) = \frac{1}{3} + \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} = \frac{z^2 + z + 1}{3z^2}$$

$$y[n] = H(z)z^n$$

num=0 $H(z) = 0$ $y[n] = 0$ $z^2 + z + 1 = 0$ **zeros**
 $z = \frac{1}{2}(-1 \pm j\sqrt{3}) = e^{\pm j2\pi/3}$ roots of numerator

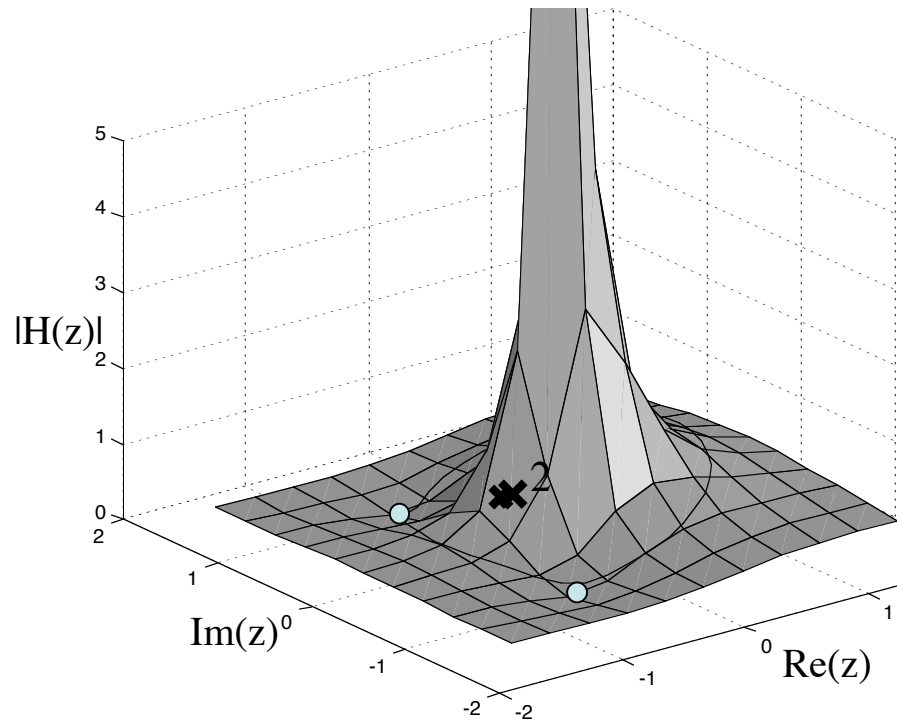
denom=0 $H(z) = \infty$ $y[n] = \infty$ $z^2 = 0$ **poles**
 $z = 0, 0$ roots of denominator



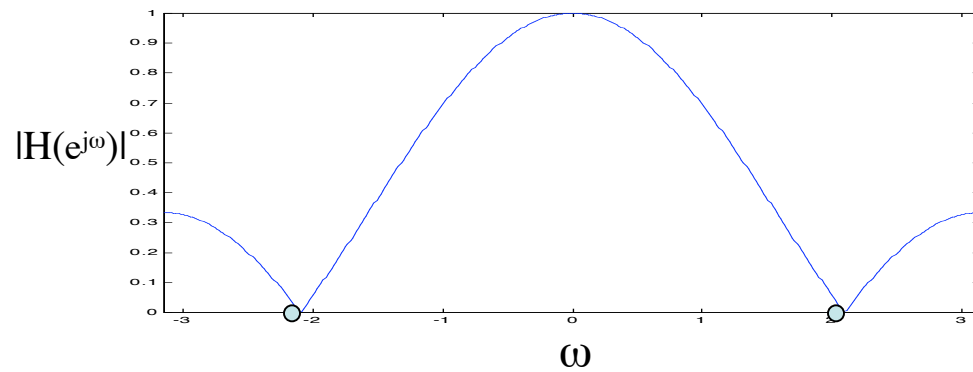
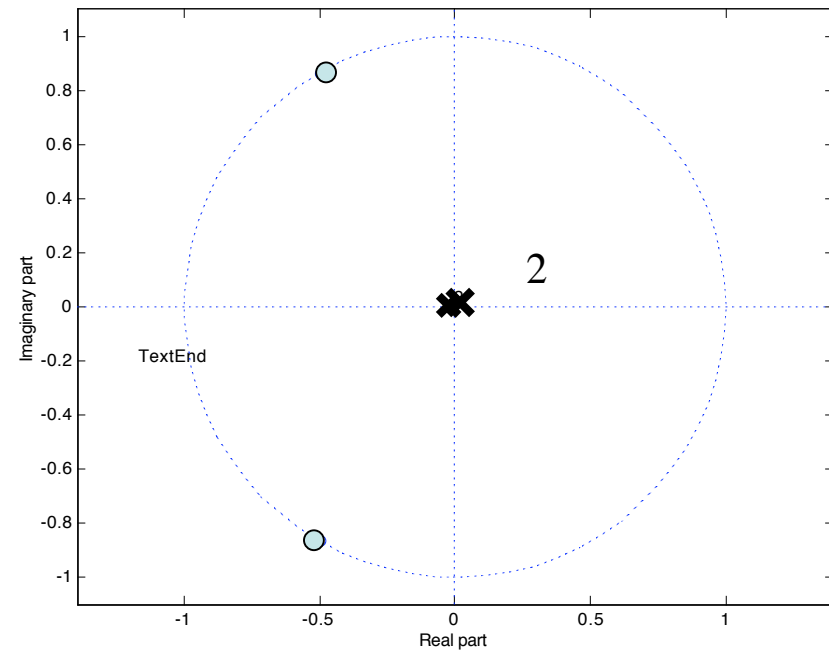
FIR filters have poles
at either at 0 or ∞ .

#poles=#zeros
“extra” zero/poles are at $z=\infty$.

system response $|H(z)|$



pz plot



frequency response

$$\mathcal{H}(\omega) = H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

The frequency response is
 $H(z)$ evaluated on unit circle

Solving impulse response

$$y[n] = x[n] - y[n-2]$$

$$x[n] = \delta[n]$$

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$

$$y[1] = x[1] - y[-1] = 0 - 0 = 0$$

$$y[2] = x[2] - y[0] = 0 - 1 = -1$$

$$y[3] = x[3] - y[1] = 0 - 0 = 0$$

$$y[4] = x[4] - y[2] = 0 - (-1) = 1$$

⋮

Remember:

$$H(z) = \frac{X(z)}{Y(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{k=1}^N a_k z^{-k}}$$

$$X(z) = \frac{z^2 - \cos(\hat{\omega})z}{z^2 - 2\cos(\hat{\omega})z + 1}$$

⇕

$$x[n] = \cos(\hat{\omega}n)u[n]$$

$$\hat{\omega} = \frac{\pi}{2} = \frac{2\pi}{4}$$

z-transform

$$Y(z) = X(z) - z^{-2}Y(z)$$

$$(1 + z^{-2})Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 + z^{-2})} \quad \text{system function}$$

$$Y(z) = H(z)X(z) = \frac{1}{(1 + z^{-2})} X(z)$$

$$Y(z) = \frac{1}{(1 + z^{-2})} = \frac{z^2}{z^2 + 1}$$

⇓ Inverse z-transform (lookup)

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n]$$

$$y[n] = \{1, 0, -1, 0, 1, \dots\}$$

Solve difference equation

$$y[n] = x[n] - y[n-2] \quad h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n] \quad \text{impulse response}$$

$$x[n] = u[n] \quad \text{step input}$$

iteration

$$y[0] = x[0] - y[-2] = 1 - 0 = 1$$

$$y[1] = x[1] - y[-1] = 1 - 0 = 1$$

$$y[2] = x[2] - y[0] = 1 - 1 = 0$$

$$y[3] = x[3] - y[1] = 1 - 1 = 0$$

$$y[4] = x[4] - y[2] = 1 - 0 = 1$$

•
•
•

convolution

x[n]	1	1	1	1	1	1	1	...
h[n]	1	0	-1	0	1	0	-1	...
=====								
	1	1	1	1	1	1	1	
		0	0	0	0	0	0	
			-1	-1	-1	-1	-1	...
				0	0	0	0	
					1	1	1	
					⋮			
=====								
	1	1	0	0	1	...		

z-transforms

$$y[n] = x[n] - y[n-2]$$

$\Downarrow z$

$$Y(z) = X(z) - z^{-2}Y(z)$$

$$Y(z) = \frac{1}{(1+z^{-2})} X(z) = H(z)X(z)$$

$$Y(z) = \frac{1}{(1+z^{-2})} \frac{1}{(1-z^{-1})}$$

$\Downarrow$ partial fraction expansion

$$Y(z) = \frac{1/2}{(1-z^{-1})} + \frac{1/2}{(1+z^{-2})} + \frac{1}{2} \frac{z^{-1}}{(1+z^{-2})}$$

$\Downarrow$ Inverse z-transform (lookup)

$$y[n] = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{2\pi}{4}n\right) + \frac{1}{2} \cos\left(\frac{2\pi}{4}(n-1)\right) \Rightarrow$$

$$x[n] = u[n]$$

$\Downarrow z$

$$X(z) = \frac{1}{1-z^{-1}}$$

multiplication

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

$$y[n] \Leftrightarrow \frac{Y(z)}{1}$$

$$a^n u[n] \Leftrightarrow \frac{1}{(1-az^{-1})}$$

$$\cos(\alpha n)u[n] \Leftrightarrow \frac{z(z-\cos\alpha)}{z^2-(2\cos\alpha)z+1}$$

$$\cos\left(\frac{2\pi}{4}n\right)u[n] \Leftrightarrow \frac{1}{(1+z^{-2})}$$

n	$y[n]$
0	$1/2+1/2+0=1$
1	$1/2+0+1/2=1$
2	$1/2-1/2+0=0$
3	$1/2+0-1/2=0$
4	$1/2+1/2+0=1$
5	$1/2+0+1/2=1$

Partial fraction expansion

$$Y(z) = \frac{1}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})} = \frac{\overset{\text{residuals}}{A}}{(1 + 2z^{-1})} + \frac{B}{(1 - \frac{3}{4}z^{-1})}$$

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

We know:

single pole @ $z=a$

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1] \quad |z| < |a|$$

complex conjugate poles

$$\frac{z(z - \gamma \cos \alpha)}{z^2 - (2\gamma \cos \alpha)z + \gamma^2} \Leftrightarrow \gamma^n \cos(\alpha n) u[n] \quad |z| > |\gamma|$$

Partial fraction expansion

$$Y(z) = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)} = \frac{A}{\left(1 + 2z^{-1}\right)} + \frac{B}{\left(1 - \frac{3}{4}z^{-1}\right)}$$

Sum of responses of individual single real poles, or complex conjugate pairs of poles.

⇓ cross multiply

$$Y(z) = \frac{A\left(1 - \frac{3}{4}z^{-1}\right) + B\left(1 + 2z^{-1}\right)}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

⇓ collect terms

$$Y(z) = \frac{\left(-\frac{3}{4}A + 2B\right)z^{-1} + (A + B)}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n - 1] \quad |z| < |a|$$

$$\begin{aligned}
 Y(z) &= \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)} \\
 &= \frac{A}{\left(1 + 2z^{-1}\right)} + \frac{B}{\left(1 - \frac{3}{4}z^{-1}\right)} \\
 &= \frac{\left(-\frac{3}{4}A + 2B\right)z^{-1} + (A + B)}{\left(1 + 2z^{-1}\right)\left(1 - z^{-1}\right)}
 \end{aligned}$$

$$-\frac{3}{4}A + 2B = 0 \quad A + B = 1 \quad \text{match coefficients}$$

$$A = \frac{8}{11} \quad B = \frac{3}{11}$$

$$Y(z) = \frac{8/11}{\left(1 + 2z^{-1}\right)} + \frac{3/11}{\left(1 - \frac{3}{4}z^{-1}\right)} = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n - 1] \quad |z| < |a|$$

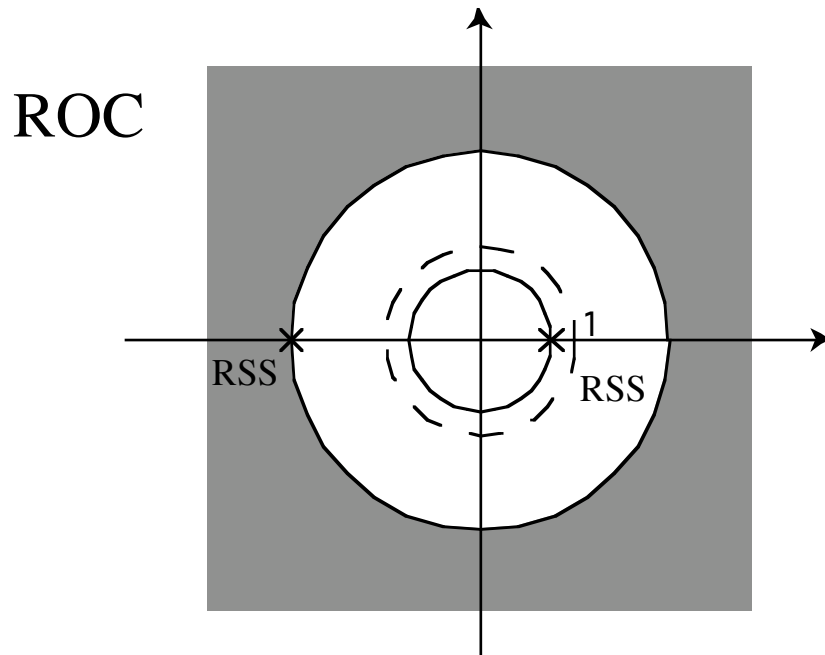
Inverse z-transform

$$Y(z) = \frac{1}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})}$$

$$= \frac{8/11}{(1 + 2z^{-1})} + \frac{3/11}{(1 - \frac{3}{4}z^{-1})} \quad \text{poles: } 3/4, -2$$

$$y[n] = \frac{8}{11}(-2)^n u[n] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \quad |z| > |2|$$

causal, unstable



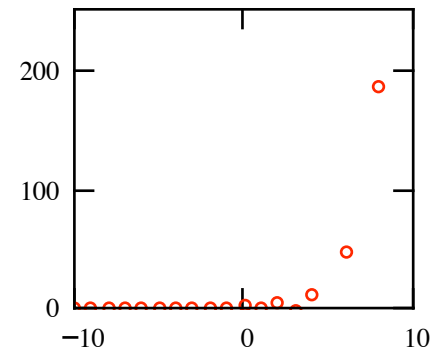
We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1] \quad |z| < |a|$$



Inverse z-transform

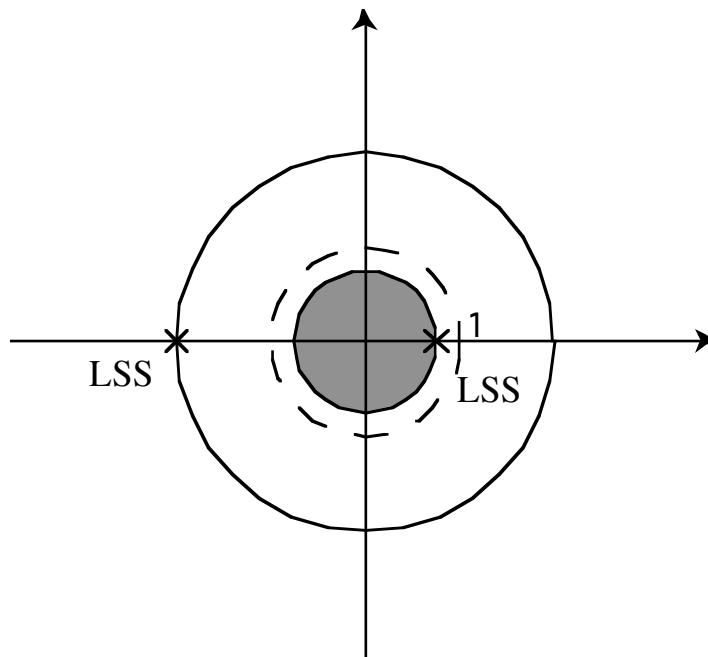
$$Y(z) = \frac{1}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})}$$

$$= \frac{8/11}{(1 + 2z^{-1})} + \frac{3/11}{(1 - \frac{3}{4}z^{-1})} \quad \text{poles: } 3/4, -2$$

$$\Downarrow \text{Inverse z-transform}$$

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \quad |z| < \left|\frac{3}{4}\right|$$

anticausal, unstable



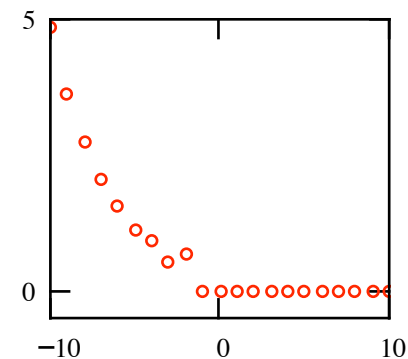
We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1] \quad |z| < |a|$$



Inverse z-transform

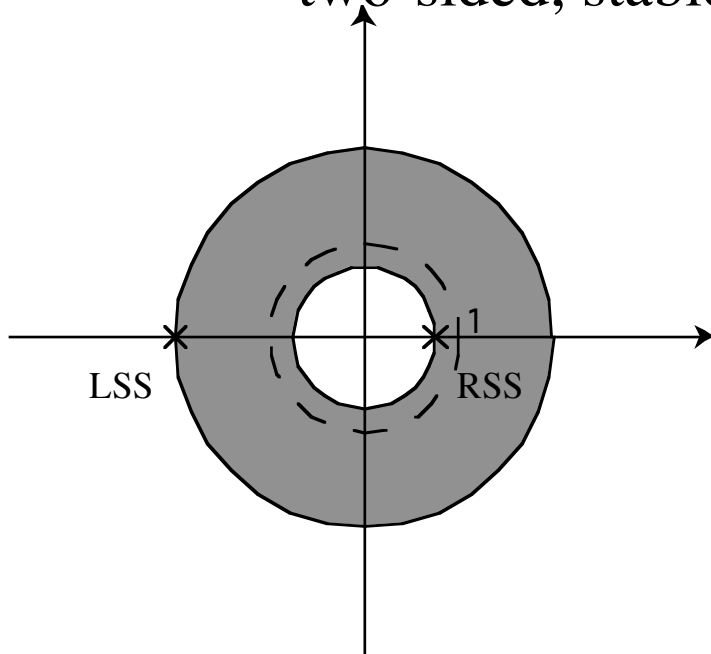
$$Y(z) = \frac{1}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})}$$

$$= \frac{8/11}{(1 + 2z^{-1})} + \frac{3/11}{(1 - \frac{3}{4}z^{-1})} \quad \text{poles: } 3/4, -2$$

↓ Inverse z-transform

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \quad \left|\frac{3}{4}\right| < |z| < |2|$$

two-sided, stable



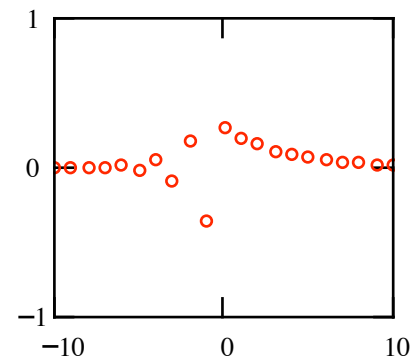
We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1] \quad |z| < |a|$$



Inverse z-transform

$$Y(z) = \frac{1}{\left(1 + 2z^{-1}\right)\left(1 - \frac{3}{4}z^{-1}\right)}$$

$$= \frac{8/11}{\left(1 + 2z^{-1}\right)} + \frac{3/11}{\left(1 - \frac{3}{4}z^{-1}\right)} \quad \text{poles: } 3/4, -2$$

↓ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \quad \left|\frac{3}{4}\right| < |z| \cap |z| > 2$$

not possible

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow a^n u[n] \quad |z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} \Leftrightarrow -a^n u[-n-1] \quad |z| < |a|$$

Inverse z-transform

$$Y(z) = \frac{1}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})}$$

$$= \frac{8/11}{(1 + 2z^{-1})} + \frac{3/11}{(1 - \frac{3}{4}z^{-1})} \quad \text{poles: } 3/4, -2$$

↓ Inverse z-transform

$$y[n] = \frac{8}{11}(-2)^n u[n] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \quad |z| > |2| \quad \text{causal, unstable}$$

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \quad |z| < \left|\frac{3}{4}\right| \quad \text{anticausal, unstable}$$

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n] \quad \left|\frac{3}{4}\right| < |z| < |2| \quad \text{two-sided, stable}$$

$$y[n] = \frac{8}{11}(-2)^n u[n] - \frac{3}{11}\left(\frac{3}{4}\right)^n u[-n-1] \quad \left|\frac{3}{4}\right| < |z| \cap |z| > |2| \quad \text{not possible}$$

Partial fraction expansion II

$$Y(z) = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} \cdot \frac{z^{-1}}{z^{-1}} = \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4}$$

Must be in terms only of z^{-1}

$$\begin{aligned} Y(z) &= \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4} = \frac{4 + 7.6z^{-1}}{(1 + 2z^{-1})(4 - 3z^{-1})} && \text{factor denominator into form} \\ & && (1 + p_1z^{-1})(1 - p_2z^{-1}) \\ &= \frac{1 + 1.9z^{-1}}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})} \\ &= \frac{A}{(1 + 2z^{-1})} + \frac{B}{(1 - \frac{3}{4}z^{-1})} \end{aligned}$$

How to find A,B?

“coverup” method

$$Y(z) \cdot (1 + 2z^{-1}) = \frac{A}{(1 + 2z^{-1})} \cdot (1 + 2z^{-1}) + \frac{B}{(1 - \frac{3}{4}z^{-1})} \cdot (1 + 2z^{-1})$$

$$Y(z) \cdot (1 + 2z^{-1}) = A + \frac{B}{(1 - \frac{3}{4}z^{-1})} \cdot (1 + 2z^{-1})$$

$$Y(z) \cdot (1 + 2z^{-1}) \Big|_{z=-2} = A + \frac{B}{(1 - \frac{3}{4}z^{-1})} \cdot 0 = A$$

Partial fraction expansion II

$$\begin{aligned} Y(z) &= \frac{4 + 7.6z^{-1}}{-6z^{-2} + 5z^{-1} + 4} = \frac{4 + 7.6z^{-1}}{(1 + 2z^{-1})(4 - 3z^{-1})} = \frac{1 + 1.9z^{-1}}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})} \\ &= \frac{A}{(1 + 2z^{-1})} + \frac{B}{(1 - \frac{3}{4}z^{-1})} \end{aligned}$$

$$A = Y(z)(1 + 2z^{-1}) \Big|_{z=-2} = \frac{1 + 1.9z^{-1}}{(1 - \frac{3}{4}z^{-1})} \Big|_{z=-2} = 0.036$$

$$B = Y(z)(1 - \frac{3}{4}z^{-1}) \Big|_{z=\frac{3}{4}} = \frac{1 + 1.9z^{-1}}{(1 + 2z^{-1})} \Big|_{z=\frac{3}{4}} = 0.964$$

$$Y(z) = \frac{0.036}{(1 + 2z^{-1})} + \frac{0.964}{(1 - \frac{3}{4}z^{-1})}$$

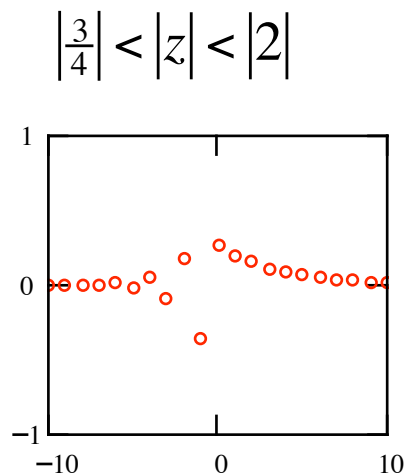
Effects of a zero

zeros: 0,0

$$Y(z) = \frac{1}{(1+2z^{-1})(1-\frac{3}{4}z^{-1})} \quad \text{poles: } 3/4, -2$$

$$= \frac{0.73}{(1+2z^{-1})} + \frac{0.27}{(1-\frac{3}{4}z^{-1})}$$

$$y[n] = -\frac{8}{11}(-2)^n u[-n-1] + \frac{3}{11}\left(\frac{3}{4}\right)^n u[n]$$



zero at $z=-1.9$
close to pole at $z=-2$,
pole's effect reduced
(0.036 vs. 0.727)

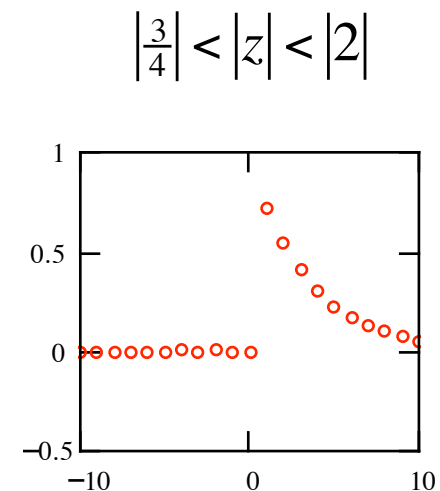
two sided sequences

zeros: 0,-1.9

$$Y(z) = \frac{1+1.9z^{-1}}{(1+2z^{-1})(1-\frac{3}{4}z^{-1})}$$

$$Y(z) = \frac{0.036}{(1+2z^{-1})} + \frac{0.964}{(1-\frac{3}{4}z^{-1})}$$

$$y[n] = -0.036(-2)^n u[-n-1] + 0.964\left(\frac{3}{4}\right)^n u[n]$$



Partial fraction expansion III

$$Y(z) = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} = \frac{4z + 7.6}{-6z^{-1} + 5 + 4z} \cdot \frac{z}{z}$$

Must be in terms only of z

$$= \frac{4z^2 + 7.6z}{4z^2 + 5z - 6}$$

$$= \frac{z^2 + 1.9z}{(z + 2)(z - \frac{3}{4})}$$

factor into form $(z-p1)(z-p2)$

Hint: use matlab's **root** command

$$= \frac{Az}{(z + 2)} + \frac{Bz}{(z - \frac{3}{4})} + C$$

We know:

right sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

left sided sequence

$$\frac{1}{1 - az^{-1}} = \frac{z}{z - a} \Leftrightarrow -a^n u[-n - 1]$$

$$|z| < |a|$$

Partial fraction expansion III

$$Y(z) = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6} = \frac{z^2 + 1.9z}{(z+2)(z-\frac{3}{4})}$$

$$= \frac{Az}{(z+2)} + \frac{Bz}{(z-\frac{3}{4})} + C$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

$$C = Y(z)\Big|_{z=0} = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6}\Big|_{z=0} = 0$$

left sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$$B = \frac{Y(z)(z-\frac{3}{4})}{z}\Big|_{z=\frac{3}{4}} = \frac{z^2 + 1.9z}{z(z+2)}\Big|_{z=\frac{3}{4}} = 0.964$$

$$A = \frac{Y(z)(z+2)}{z}\Big|_{z=-2} = \frac{z^2 + 1.9z}{z(z-\frac{3}{4})}\Big|_{z=-2} = 0.036$$

Partial fraction expansion III

$$Y(z) = \frac{4z^2 + 7.6z}{4z^2 + 5z - 6} = \frac{z^2 + 1.9z}{(z+2)(z-\frac{3}{4})}$$

$$= \frac{0.036z}{(z+2)} + \frac{0.964z}{(z-\frac{3}{4})}$$

We know:

right sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow a^n u[n]$$

$$|z| > |a|$$

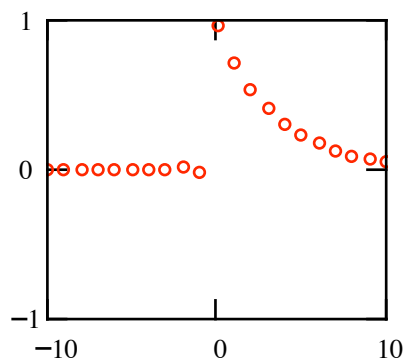
left sided sequence

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \Leftrightarrow -a^n u[-n-1]$$

$$|z| < |a|$$

$$y[n] = -0.036(-2)^n u[-n-1] + 0.964\left(\frac{3}{4}\right)^n u[n]$$

$$\left|\frac{3}{4}\right| < |z| < |2|$$



two sided sequence

Long Division

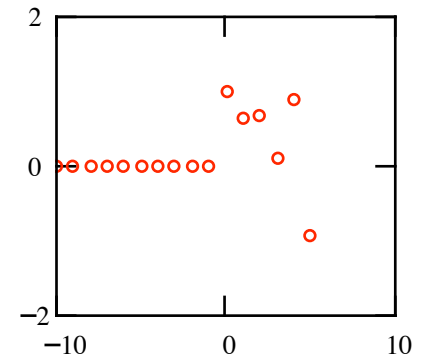
$$Y(z) = \frac{1 + 1.9z^{-1}}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})} = \frac{1 + 1.9z^{-1}}{-\frac{3}{2}z^{-2} + \frac{5}{4}z^{-1} + 1}$$

$$\begin{array}{r}
 1 + 0.65z^{-1} + 0.687z^{-2} + 0.116z^{-3} \quad \text{right sided sequence} \\
 \hline
 = 1 + \frac{5}{4}z^{-1} - \frac{3}{2}z^{-2} \bigg) 1 + 1.9z^{-1} \\
 \underline{1 + 1.25z^{-1} - 1.5z^{-2}} \\
 0.65z^{-1} + 1.5z^{-2} \\
 \underline{0.65z^{-1} + 0.813z^{-2} - 0.975z^{-3}} \\
 0.687z^{-2} + 0.975z^{-3} \\
 \underline{0.687z^{-2} + 0.859z^{-3} - 1.031z^{-4}} \\
 0.116z^{-3} + 1.031z^{-4}
 \end{array}$$

$$y[n] = \delta[n] + 0.65\delta[n-1] + 0.69\delta[n-2] + 0.12\delta[n-3] + \dots$$

compare

$$y[n] = 0.036(-2)^n u[n] + \left(\frac{3}{4}\right)^n 0.964u[n] \quad = \{1, 0.65, 0.69, 0.12\} \quad n=0,1,2,3$$



Long Division

$$Y(z) = \frac{1 + 1.9z^{-1}}{(1 + 2z^{-1})(1 - \frac{3}{4}z^{-1})} = \frac{1 + 1.9z^{-1}}{-\frac{3}{2}z^{-2} + \frac{5}{4}z^{-1} + 1}$$

left sided sequence

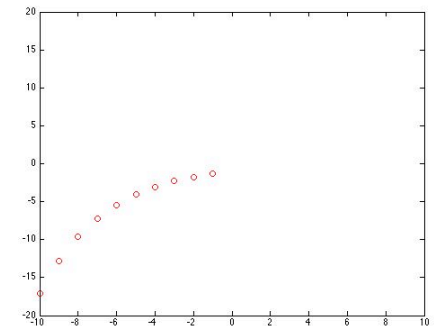
$$\begin{array}{r}
 -1.267z - 1.723z^2 - 2.281z^3 + \dots \\
 \hline
 = -1.5z^{-2} + 1.25z^{-1} + 1 \bigg) 1.9z^{-1} + 1 \\
 \underline{1.9z^{-1} - 1.584 - 1.267z} \\
 2.584 + 1.267z \\
 \underline{2.584 - 2.154z - 1.723z^{-3}} \\
 3.421z^{-2} + 0.975z^{-3} \\
 \underline{3.421z^{-2} - 2.851z^{-3} - 2.281z^{-4}}
 \end{array}$$

$$y[n] = -1.27\delta[n+1] - 1.72\delta[n+2] - 2.28\delta[n+3] + \dots$$

compare

$$y[n] = -0.036(-2)^n u[-n-1] - \left(\frac{3}{4}\right)^n 0.964 u[-n-1] = \{-1.27, -1.72, -2.28\}$$

$n = \{-1, -2, -3\}$



Fourier Transforms

Compute spectrum of signals

Fourier Series	$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j 2\pi k t / T_0} dt$	Periodic in (cont.) time Discrete freq
DTFT	$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{j\hat{\omega}k}$	Discrete time Periodic in (cont.) freq
DFT	$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k / N)n}$	Discrete & periodic time Discrete & periodic freq

Discrete Fourier Transform (DFT)

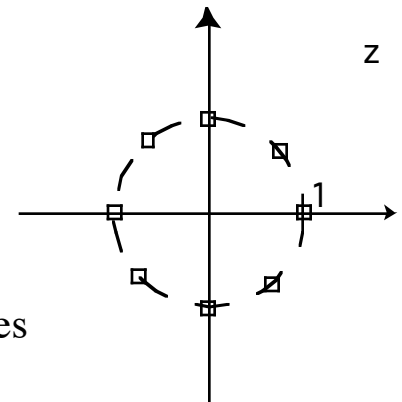
Compute spectrum of discrete-time periodic signals

N samples in time domain $\xrightleftharpoons[\text{IDFT}]{\text{DFT}}$ N complex numbers in frequency domain

DFT $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$ analysis

IDFT $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi k/N)n}$ synthesis

DFT: sample continuous $H(\omega)$ (DTFT) at N evenly spaced frequencies



$x[n]$ periodic with period N samples

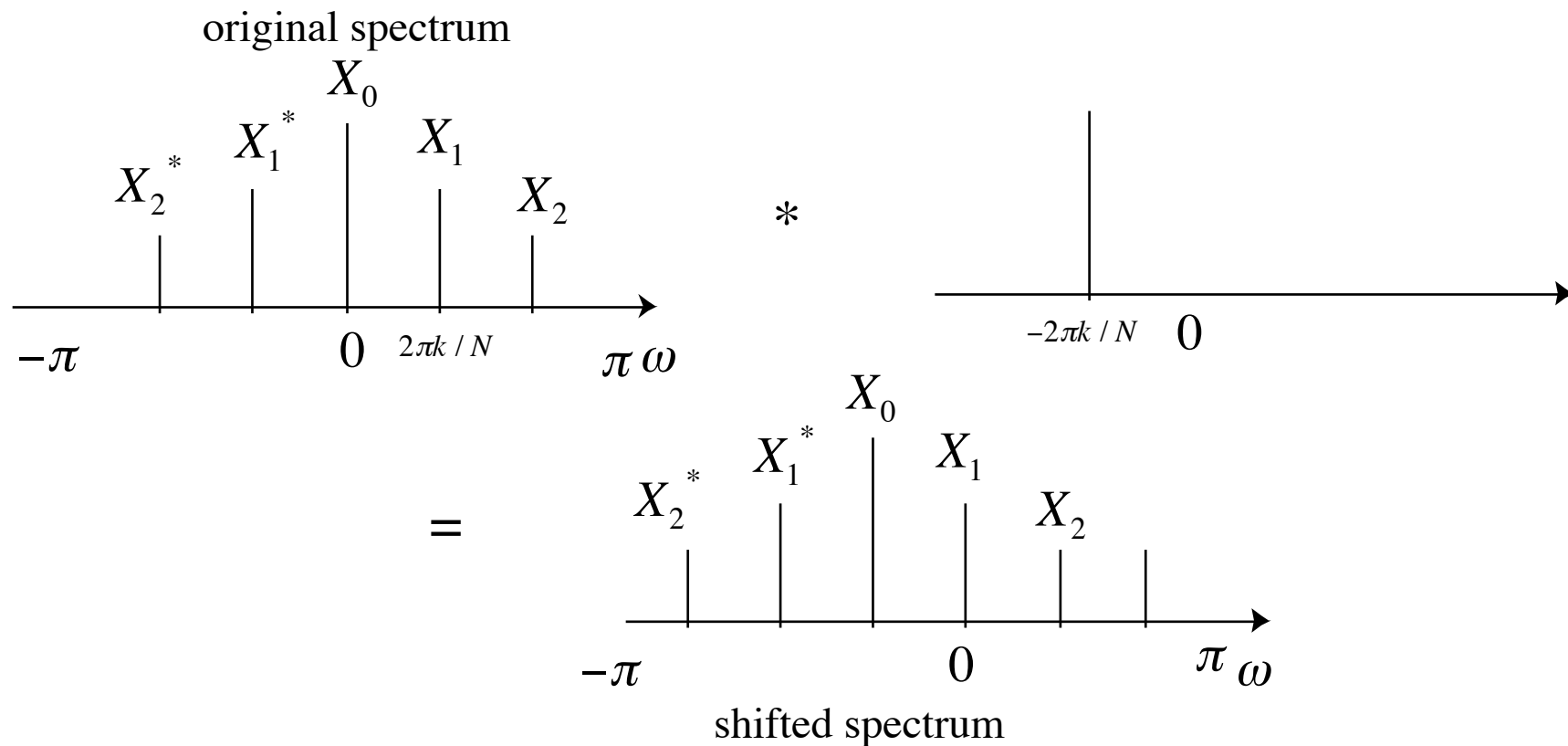
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{-j(2\pi k/N)n}$$

composed of N frequencies
harmonically related

DFT

$$X[k] = \sum_{n=0}^{N-1} \underbrace{x[n] e^{-j(2\pi k/N)n}}$$

move X_k to DC

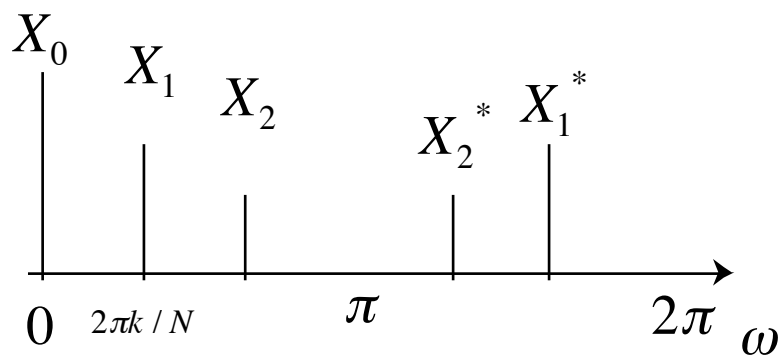


$x[n]$ periodic

DFT
$$X[k] = \sum_{n=0}^{N-1} \underbrace{x[n] e^{-j(2\pi k/N)n}}_{\text{move } X_k \text{ to DC}}$$

move X_k to DC

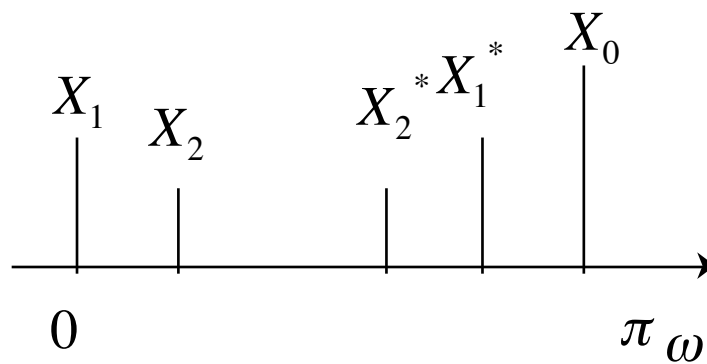
original (aliased) spectrum



*



=

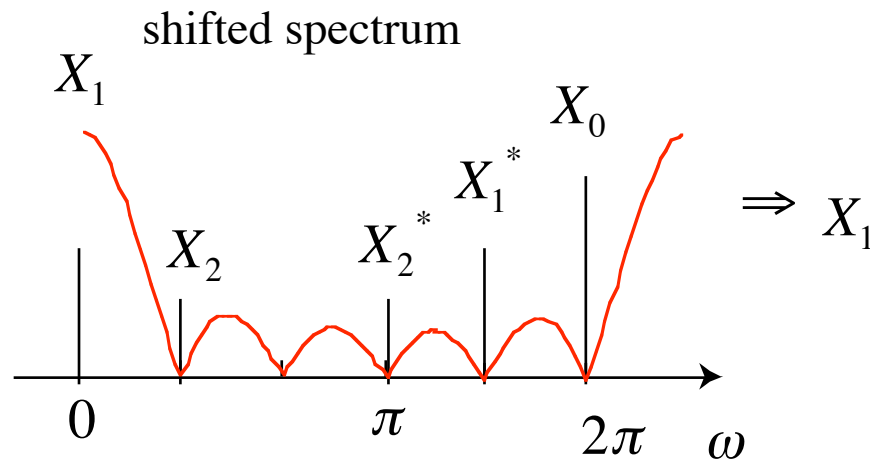


$x[n]$ periodic

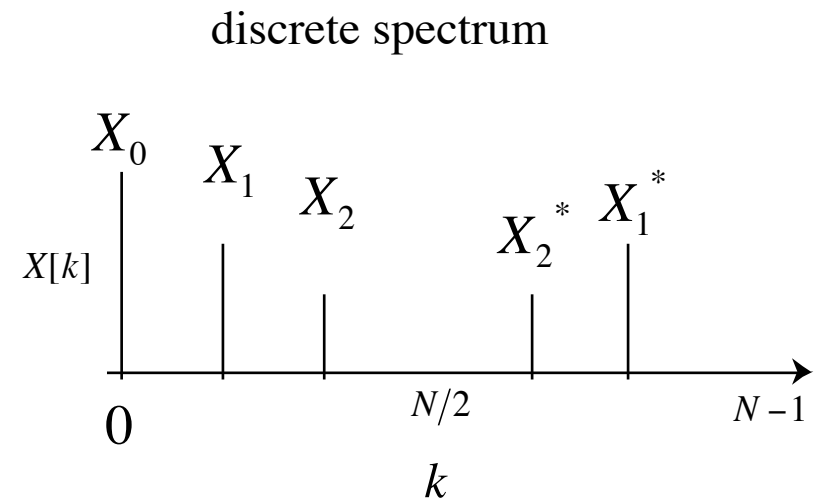
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n}$$

FIR low pass filter to measure DC
(N point running sum, $\{1,1,1\dots 1\}$)
zeros @ harmonics



FIR filter w/ zeros
at harmonics



$x[n] = \{1, 1, 1, 0\}$ Impulse response of 3pt summer

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^3 x[n] e^{-j(2\pi k/4)n}$$

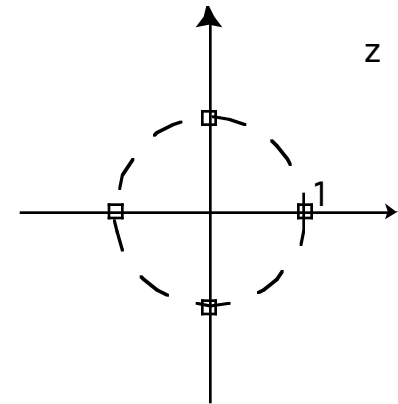
$$= x[0] + x[1]e^{-j(2\pi k/4)} + x[2]e^{-j(2\pi k/4)2} + x[3]e^{-j(2\pi k/4)3}$$

$$= 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

$$= 1 + e^{-j\frac{3\pi}{4}k} \left(e^{j\frac{\pi}{4}k} + e^{-j\frac{\pi}{4}k} \right)$$

$$= 1 + 2 \left[-\sqrt{2}(1+j) \right]^k \cos\left(\frac{\pi}{4}k\right)$$

$$X[k] = \begin{matrix} k=0, & 1, & 2, & 3 \\ \{3, & -i, & 1, & i\} \\ DC & e^{-j(\pi/2)} & e^{-j(\pi)} & e^{-j(\frac{3\pi}{2})} \end{matrix}$$



check:

```
»x=[1 1 1 0];
```

```
»X=fft(x)
```

```
X =
```

```
3.0000 0 - 1.0000i 1.0000 0 + 1.0000i
```

```
»fftshift(X)
```

```
ans =
```

```
1.0000 0 + 1.0000i 3.0000 0 - 1.0000i
```

Note: $0 < \omega < 2\pi$

only extract limited number of frequencies due to N samples per period

$$x[n] = \{1, 1, 1, 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^3 x[n] e^{-j(2\pi k/4)n}$$

$$= 1 + 2 \left[-\sqrt{2}(1+j) \right]^k \cos\left(\frac{\pi}{4}k\right)$$

$$X[k] = \{3, -i, 1, i\}$$

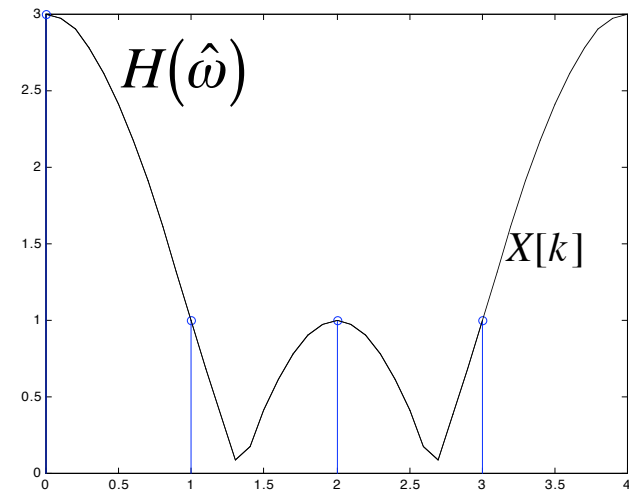
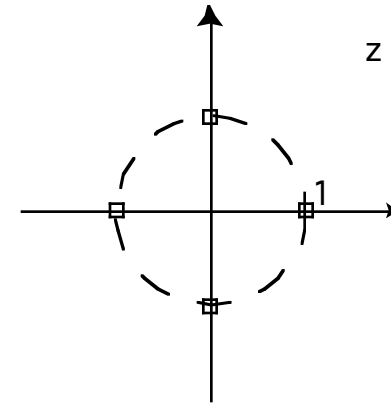
$$\begin{matrix} DC & e^{-j(\pi/2)} & e^{-j(\pi)} & e^{-j(\frac{3\pi}{2})} \end{matrix}$$

$$\begin{matrix} DC & e^{-j(\pi/2)} & e^{-j(\frac{-3\pi}{2})} & e^{-j(\frac{-\pi}{2})} \end{matrix}$$

$$H(\hat{\omega}) = e^{-j\hat{\omega}} (1 + 2\cos\hat{\omega})$$

Note: $0 < \omega < 2\pi$

only extract limited number of frequencies due to N samples per period



Redo like homework

$$x[n] = \{1, 1, 1, 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^3 x[n] e^{-j(2\pi k/4)n} = \sum_{n=0}^2 e^{-j(2\pi k/4)n}$$

$$\begin{aligned} &= \frac{1 - e^{-j(2\pi k/4)3}}{1 - e^{-j(2\pi k/4)}} = \frac{1 - e^{-j3\pi k/2}}{1 - e^{-j\pi k/2}} \\ &= \frac{1 - j^k}{1 - (-j)^k} \quad k \neq 0 \end{aligned}$$

$$X[k] = \sum_{n=0}^2 e^{-j(2\pi 0/4)n} = \sum_{n=0}^2 e^{-j0} = \sum_{n=0}^2 1 = 3$$

$$\begin{aligned} X[k] &= \{3, -j, 1, j\} \\ k &= \{0, 1, 2, 3\} \end{aligned}$$

works okay if you have
x[n]'s = 1 or complex exponentials

Remember

$$\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a}$$

k=0

$$X[1] = \frac{1-j}{1+j} \frac{1-j}{1-j} = \frac{1-2j-1}{2} = -j$$

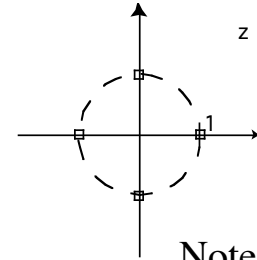
$$X[2] = \frac{1-j^2}{1-(-j)^2} = \frac{1-(-1)}{1-(-1)} = 1$$

$$X[3] = \frac{1-j^3}{1-(-j)^3} = \frac{1-(-j)}{1-j} = \frac{1+j}{1-j} \frac{1+j}{1+j} = \frac{1+2j-1}{1+2} = j$$

Padding

$$x[n] = \{1, 1, 1, 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^3 x[n] e^{-j(2\pi k/4)n} = \sum_{n=0}^3 x[n] e^{-j(\pi/2)kn}$$

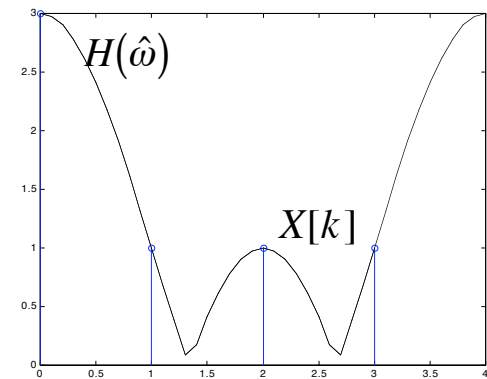


Note: $0 < \omega < 2\pi$

$$X[k] = \begin{matrix} k=0, & 1, & 2, & 3 \\ \{3, & -i, & 1, & i\} \\ \text{DC} & \pi/2 & \pi & \frac{3\pi}{2} \\ \text{DC} & \pi/2 & \pi & -\frac{\pi}{2} \end{matrix}$$

$X[k]$ is sampled
version of $H(\hat{\omega})$

N time samples =
N frequency samples



$$H(\hat{\omega}) = e^{-j\omega} (1 + 2\cos\hat{\omega})$$

Padding

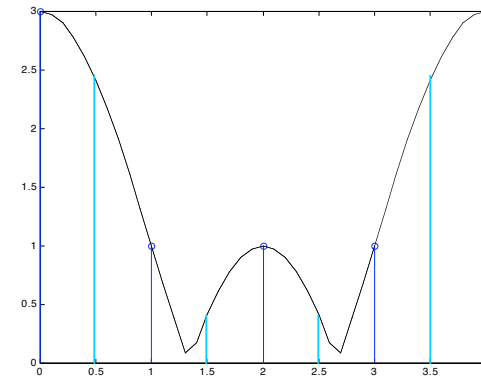
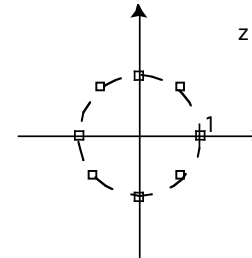
Pad $x[n]$ to get more samples of $H(\hat{\omega})$

$$x[n] = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^7 x[n] e^{-j(2\pi k/8)n} = \sum_{n=0}^2 x[n] e^{-j(2\pi k/8)n}$$

$$= x[0] + x[1]e^{-j(2\pi k/8)} + x[2]e^{-j(2\pi 2k/8)}$$

	k=0,	1,		2,	3,		4,		5,		6,		7
$X[k] = \{$	3	$1.7 - j1.7$	$-i$	$.29 + j2.9$	1	$0.29 - j0.29$	i	$1.7 + j1.7$	$\}$				
	DC	$\pi/4$	$\pi/2$	$3\pi/4$	π	$-3\pi/4$	$-\pi/2$	$-\pi/4$					



$$H(\hat{\omega}) = e^{-j\hat{\omega}}(1 + 2\cos\hat{\omega})$$

DFT Convolution

$$y[n] * x[n] \xleftrightarrow{\text{DTFT}} Z(\hat{\omega}) = Y(\hat{\omega})X(\hat{\omega}) \xleftrightarrow{\text{IDTFT}} z[n]$$

sample
domain

frequency
domain

sample
domain

$Y[k]$ sampled version of $Y(\hat{\omega})$

Use DFT to compute $Y[k]$ and $X[k]$

$$y[n] \otimes x[n] \xleftrightarrow{\text{DFT}} Z[k] = Y[k]X[k] \xleftrightarrow{\text{IDFT}} z[n]$$

circular
convolution

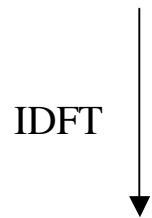
DFT Convolution

Problem:

$$z[n] = y[n] * x[n]$$



$$Z[k] = Y[k]X[k]$$



$$z'[n]$$

$x[n]$: length N

$y[n]$: length N

$z[n]$: length $N+N-1$

$X[k]$: length N

$Y[k]$: length N

$Z[k]$: length N

$z'[n]$: length N

undersampled frequency



aliased time samples

DFT Convolution

ex.

$$x[n]=[1 \ -1 \ 1], \ y[n]=[1 \ 2 \ 3]$$

$$z'[n]=x[n]*y[n]=[1 \ 1 \ 2 \ -1 \ 3]$$

$$\begin{array}{r} x[n] \ 1 \ -1 \ 1 \\ y[n] \ 1 \ 2 \ 3 \\ \hline \\ \\ \\ \\ \\ \\ \hline \\ \\ \\ \\ \\ \hline z'[n] \ 1 \ 1 \ 2 \ -1 \ 3 \end{array}$$

3pt DFT

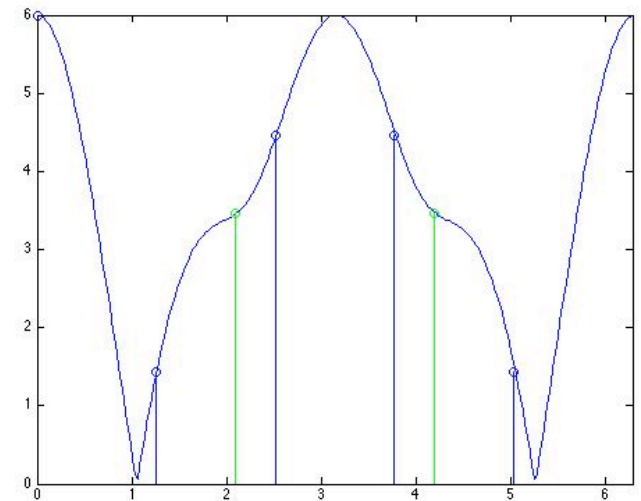
$$X[k]=[1 \ 1+j1.7321 \ 1-j1.7321]$$

$$Y[k]=[6 \ -1.5+j0.866 \ -1.5-j0.866]$$

$$Z[k]=X[k]Y[k]=[6 \ -3-j1.73 \ -3+j1.73]$$

$$z'[n]=[1 \ 1 \ 2 \ -1 \ 3]$$

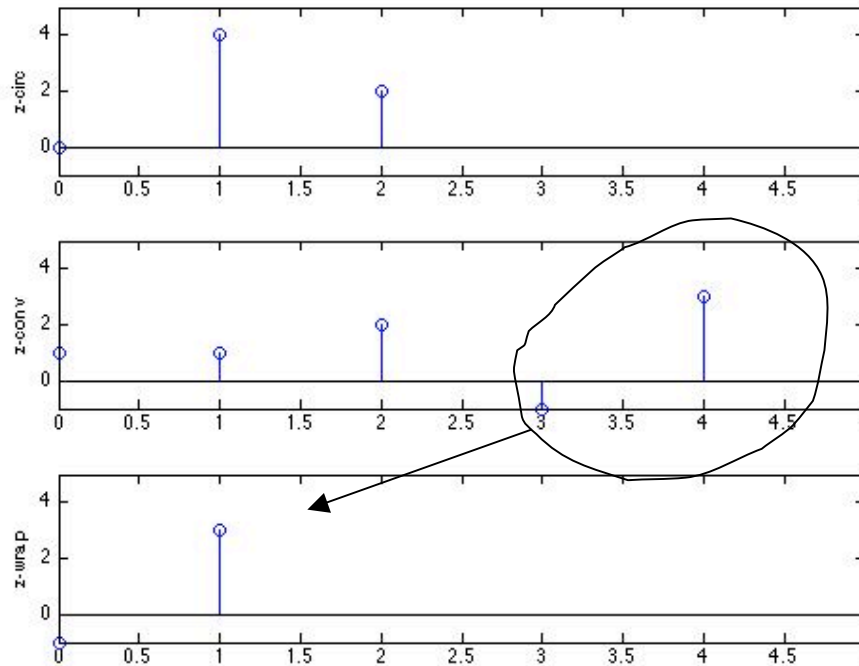
$$Z'[k]=[6 \ 1.43+j0.139 \ -1.93+j4.03 \ -1.93-j4.03 \ 1.43-j0.139]$$



$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$

↓ 3pt IDFT

$$z[k] = x[n] \otimes y[n] = [0 \quad 4 \quad 2]$$



temporal aliasing
samples wrap

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

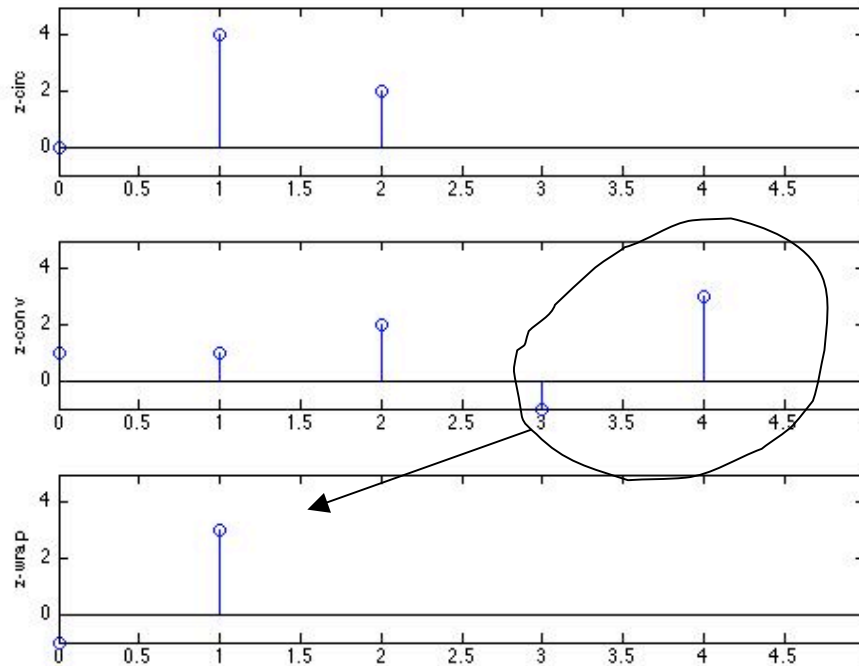
↓ 5pt IDFT

$$z'[n] = x[n] * y[n] = [1 \quad 1 \quad 2 \quad -1 \quad 3]$$

$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$

↓ 3pt IDFT

$$z[k] = x[n] \otimes y[n] = [0 \quad 4 \quad 2]$$



temporal aliasing
samples wrap

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

↓ 5pt IDFT

$$z'[n] = x[n] * y[n] = [1 \quad 1 \quad 2 \quad -1 \quad 3]$$

DFT Convolution

ex.

$x[n]=[1 \ -1 \ 1 \ 0 \ 0]$, $y[n]=[1 \ 2 \ 3 \ 0 \ 0]$ To avoid temporal aliasing, pad signals
so lengths are $2N-1$

$$X[k] = [1 \ -0.118 + j0.363 \ 2.12 + j1.54 \ 2.12 - j1.54 \ -0.118 - j0.363]$$

$$Y[k] = [6 \ -0.809 - j3.67 \ 0.309 + j1.68i \ 0.309 - j1.68 \ -0.809 + j3.67]$$

$$Z[k] = [6 \ 1.43 + j0.139 \ -1.93 + j4.03 \ -1.93 - j4.03 \ 1.43 - j0.139]$$

$$z[n] = x[n] \otimes y[n] = [1 \ 1 \ 2 \ -1 \ 3]$$

If $\text{len}(x)=N$, $\text{len}(y)=M$, then pad so lengths are $N+M-1$

How does a Fast Fourier Transform (FFT) work?

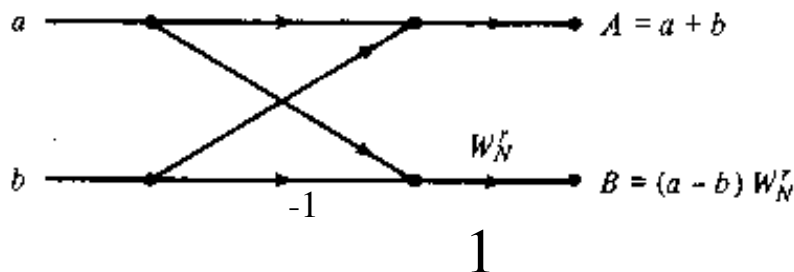
2 Point DFT

$$\begin{aligned} \text{DFT} \quad X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \quad N=2 \\ &= x[0] e^{-j(2\pi k/4)0} + x[1] e^{-j(2\pi k/2)1} \\ &= x[0] + x[1] e^{-j(\pi k)} \end{aligned}$$

FFT is an efficient way of calculating a DFT.

$$\begin{aligned} X_2[0] &= x[0] + x[1] & \begin{matrix} 1 & 1 \end{matrix} & \begin{matrix} \text{coefficients} \\ \text{block} \end{matrix} \\ X_2[1] &= x[0] - x[1] & \begin{matrix} 1 & -1 \end{matrix} \end{aligned}$$

FFT butterfly



$$\begin{aligned} W_2^k &= e^{-j(2\pi k/2)} \\ W_2^0 &= 1 \end{aligned}$$

$N^2=4$ mult
 $N^2-N=2$ adds

4 Point DFT

$$\begin{aligned}
 \text{DFT} \quad X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \quad N=4 \\
 &= x[0]e^{-j(2\pi k/4)0} + x[1]e^{-j(2\pi k/4)1} + x[2]e^{-j(2\pi k/4)2} + x[3]e^{-j(2\pi k/4)3} \\
 &= x[0] + x[1]e^{-j(\frac{\pi}{2}k)} + x[2]e^{-j(\pi k)} + x[3]e^{-j(\frac{3\pi}{2}k)}
 \end{aligned}$$

$$X[0] = x[0] + x[1] + x[2] + x[3] \quad \text{running sum}$$

$$\begin{aligned}
 X[1] &= x[0] + x[1]e^{-j(\frac{\pi}{2})} + x[2]e^{-j(\pi)} + x[3]e^{-j(\frac{3\pi}{2})} \\
 &= x[0] - jx[1] - x[2] + jx[3]
 \end{aligned}$$

$$\begin{aligned}
 X[2] &= x[0] + x[1]e^{-j(\pi)} + x[2]e^{-j(2\pi)} + x[3]e^{-j(3\pi)} \\
 &= x[0] - x[1] + x[2] - x[3]
 \end{aligned}$$

$$\begin{aligned}
 X[3] &= x[0] + x[1]e^{-j(\frac{3\pi}{2})} + x[2]e^{-j(3\pi)} + x[3]e^{-j(\frac{9\pi}{2})} \\
 &= x[0] + jx[1] - x[2] - jx[3]
 \end{aligned}$$

$$\begin{aligned}
 X[4] &= x[0] + x[1]e^{-j(2\pi)} + x[2]e^{-j(4\pi)} + x[3]e^{-j(6\pi)} \\
 &= x[0] + x[1] + x[2] + x[3] \quad \text{alias of } X[0]
 \end{aligned}$$

$N^2=16$ mult
 $N^2-N=12$ adds

4 Point DFT

$$\begin{aligned}\text{DFT} \quad X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \quad N=4 \\ &= x[0] + x[1]e^{-j(\frac{\pi}{2}k)} + x[2]e^{-j(\pi k)} + x[3]e^{-j(\frac{3\pi}{2}k)}\end{aligned}$$

$$X[0] = x[0] + x[1] + x[2] + x[3] \quad \text{running sum}$$

$$X[1] = x[0] - jx[1] - x[2] + jx[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + jx[1] - x[2] - jx[3]$$

$$X[4] = x[0] + x[1] + x[2] + x[3] \quad \text{alias of } X[0]$$

$$X[5] = x[0] - jx[1] - x[2] + jx[3] \quad \text{alias of } X[1]$$

periodic in frequency

only extract limited number of frequencies due to N samples per period

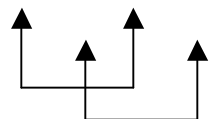
DFT

N=4

$$\begin{aligned} X_4[0] &= x[0] + x[1] + x[2] + x[3] & \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \\ X_4[1] &= x[0] - jx[1] - x[2] + jx[3] & \begin{bmatrix} 1 & -j & -1 & j \end{bmatrix} \\ X_4[2] &= x[0] - x[1] + x[2] - x[3] & \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ X_4[3] &= x[0] + jx[1] - x[2] - jx[3] & \begin{bmatrix} 1 & j & -1 & -j \end{bmatrix} \end{aligned}$$

even: X[0],X[2]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$



odd: X[1],X[3]

$$\begin{bmatrix} 1 & -j & -1 & j \\ 1 & j & -1 & -j \end{bmatrix}$$

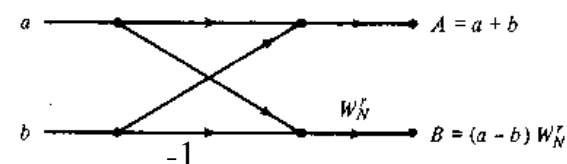
Looks like a 2pt FFT
of combined signals

Looks like a 2pt FFT
of combined signals

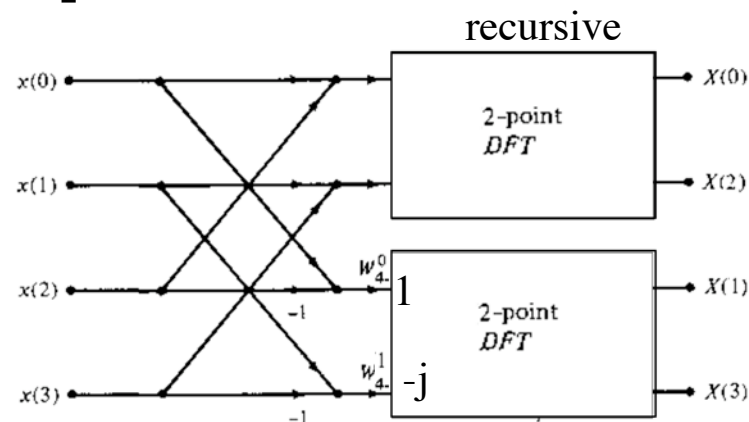
$$\begin{aligned} X_4[1] &= [x[0] - x[2]] + j[x[1] - x[3]] = X_2[0]_{\text{odd}} \\ X_4[3] &= [x[0] - x[2]] - j[x[1] - x[3]] = X_2[1]_{\text{odd}} \end{aligned}$$

N=2

$$\begin{aligned} X_2[0] &= x[0] + x[1] & \begin{bmatrix} 1 & 1 \end{bmatrix} \\ X_2[1] &= x[0] - x[1] & \begin{bmatrix} 1 & -1 \end{bmatrix} \end{aligned}$$



$$W_4^k = e^{-j(2\pi k/4)}$$



FFT

fewer multiplies/adds
 $N/2 \log_2 N = 4$ mult

DFT

N=4

$$X_4[0] = x[0] + x[1] + x[2] + x[3]$$

$$X[1] = x[0] - jx[1] - x[2] + jx[3]$$

$$X[2] = x[0] - x[1] + x[2] - x[3]$$

$$X[3] = x[0] + jx[1] - x[2] - jx[3]$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\begin{bmatrix} 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

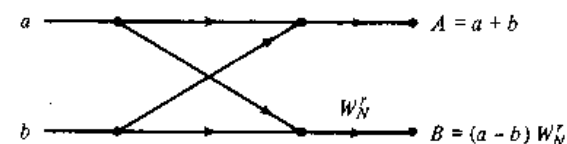
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\begin{bmatrix} 1 & j & -1 & -j \end{bmatrix}$$

N=2

$$X_2[0] = x[0] + x[1] \quad \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$X_2[1] = x[0] - x[1] \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

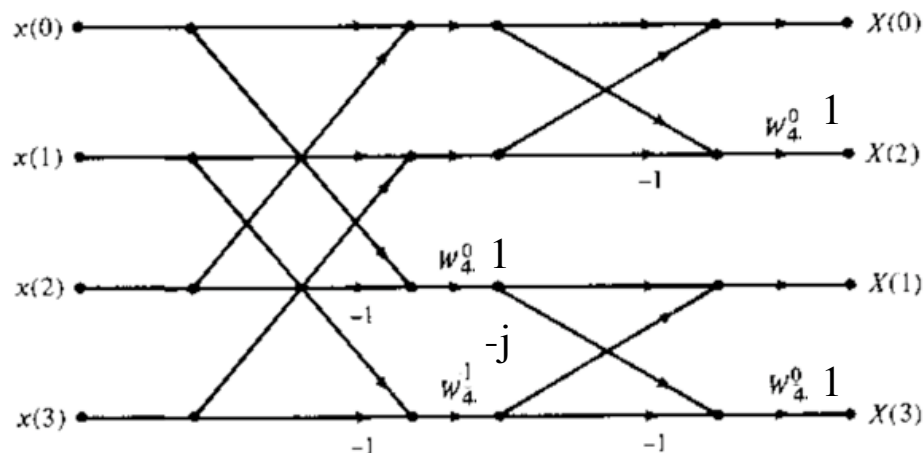


even: X[0],X[2]

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

odd: X[1],X[3]

$$\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} 1 & j \\ -1 & -j \end{bmatrix}$$



4pt FFT butterfly

$N/2 \log_2 N = 4$ mult

$N \log_2 N = 8$ add

$$\begin{aligned}
\text{DFT} \quad X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \quad N=8 \\
&= x[0]e^{-j(2\pi k/8)0} + x[1]e^{-j(2\pi k/8)1} + x[2]e^{-j(2\pi k/8)2} + x[3]e^{-j(2\pi k/8)3} \\
&\quad x[4]e^{-j(2\pi k/8)4} + x[5]e^{-j(2\pi k/8)5} + x[6]e^{-j(2\pi k/8)6} + x[7]e^{-j(2\pi k/8)7} \\
&= x[0] + x[1]e^{-j(\frac{\pi}{4}k)} + x[2]e^{-j(\frac{\pi}{2}k)} + x[3]e^{-j(\frac{3\pi}{4}k)} \\
&\quad x[4]e^{-j(\pi k)} + x[5]e^{-j(\frac{5\pi}{4}k)} + x[6]e^{-j(\frac{6\pi}{4}k)} + x[7]e^{-j(\frac{7\pi}{4}k)}
\end{aligned}$$

$$X[0] = x[0] + x[1] + x[2] + x[3] + x[4] + x[5] + x[6] + x[7] \quad \text{running sum}$$

$$\begin{aligned}
X[1] &= x[0] + \frac{\sqrt{2}}{2}(1-j)x[1] - jx[2] - \frac{\sqrt{2}}{2}(1+j)x[3] \\
&\quad -x[4] + \frac{\sqrt{2}}{2}(-1+j)x[5] + jx[6] + \frac{\sqrt{2}}{2}(1+j)x[7]
\end{aligned}$$

$$X[2] = x[0] - jx[1] - x[2] + jx[3] + x[4] - jx[5] - x[6] + jx[7]$$

$$\begin{aligned}
X[3] &= x[0] + \frac{\sqrt{2}}{2}(-1-j)x[1] + jx[2] + \frac{\sqrt{2}}{2}(1-j)x[3] \\
&\quad -x[4] + \frac{\sqrt{2}}{2}(1+j)x[5] - jx[6] + \frac{\sqrt{2}}{2}(-1+j)x[7]
\end{aligned}$$

⋮

$N^2=64$ mult

$N^2-N=56$ add

$$\text{DFT} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} \quad N=8$$

$$= x[0] + x[1]e^{-j(\frac{\pi}{4}k)} + x[2]e^{-j(\frac{\pi}{2}k)} + x[3]e^{-j(\frac{3\pi}{4}k)} \\ x[4]e^{-j(\pi k)} + x[5]e^{-j(\frac{5\pi}{4}k)} + x[6]e^{-j(\frac{6\pi}{4}k)} + x[7]e^{-j(\frac{7\pi}{4}k)}$$

$x[n]$

$X[k]$	1	1	1	1	1	1	1	1
	1	$\frac{\sqrt{2}}{2}(1-j)$	$-j$	$\frac{\sqrt{2}}{2}(-1-j)$	-1	$-\frac{\sqrt{2}}{2}(1-j)$	j	$\frac{\sqrt{2}}{2}(1+j)$
	1	$-j$	-1	j	1	$-j$	-1	j
	1	$\frac{\sqrt{2}}{2}(-1-j)$	j	$\frac{\sqrt{2}}{2}(1-j)$	-1	$-\frac{\sqrt{2}}{2}(-1-j)$	$-j$	$\frac{\sqrt{2}}{2}(-1+j)$
	1	-1	1	-1	1	-1	1	-1
	1	$\frac{\sqrt{2}}{2}(-1+j)$	$-j$	$\frac{\sqrt{2}}{2}(1+j)$	-1	$-\frac{\sqrt{2}}{2}(-1+j)$	j	$\frac{\sqrt{2}}{2}(-1-j)$
	1	j	-1	$-j$	1	j	-1	$-j$
	1	$\frac{\sqrt{2}}{2}(1+j)$	j	$\frac{\sqrt{2}}{2}(-1+j)$	-1	$-\frac{\sqrt{2}}{2}(1+j)$	$-j$	$\frac{\sqrt{2}}{2}(1-j)$

$N^2=64$ mult

$N^2-N=56$ add

even: X[0],X[2],X[4],X[6]

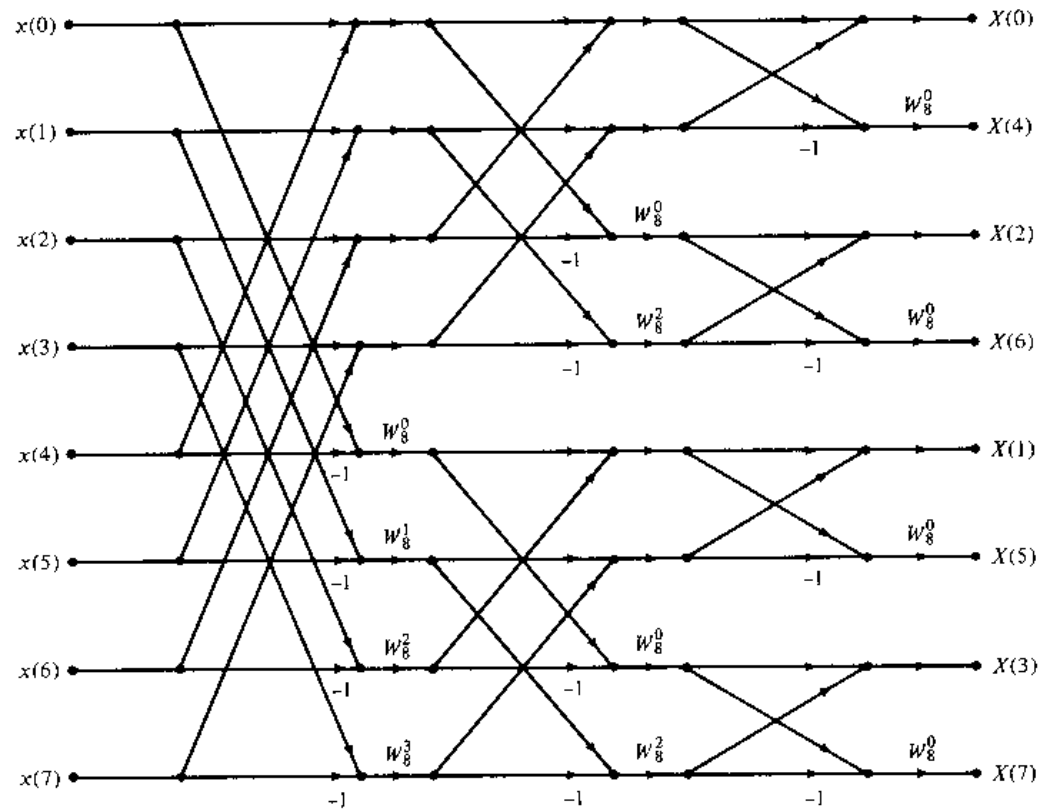
N=4

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

odd: X[1],X[3],X[5],X[7]

$$\begin{array}{cccc|cccc} 1 & \frac{\sqrt{2}}{2}(1-j) & -\mathbf{j} & \frac{\sqrt{2}}{2}(-1-j) & -1 & -\frac{\sqrt{2}}{2}(1-j) & \mathbf{j} & \frac{\sqrt{2}}{2}(1+j) \\ 1 & \frac{\sqrt{2}}{2}(-1-j) & \mathbf{j} & \frac{\sqrt{2}}{2}(1-j) & -1 & -\frac{\sqrt{2}}{2}(-1-j) & -\mathbf{j} & \frac{\sqrt{2}}{2}(-1+j) \\ 1 & \frac{\sqrt{2}}{2}(-1+j) & -\mathbf{j} & \frac{\sqrt{2}}{2}(1+j) & -1 & -\frac{\sqrt{2}}{2}(-1+j) & \mathbf{j} & \frac{\sqrt{2}}{2}(-1-j) \\ 1 & \frac{\sqrt{2}}{2}(1+j) & \mathbf{j} & \frac{\sqrt{2}}{2}(-1+j) & -1 & -\frac{\sqrt{2}}{2}(1+j) & -\mathbf{j} & \frac{\sqrt{2}}{2}(1-j) \end{array}$$

8pt FFT



$$W_8^k = e^{-j(2\pi k/8)}$$

$N/2 \log_2 N = 12$ mult
 $N \log_2 N = 24$ adds

$$y[n] = x[n] - y[n-2]$$

↓ z-transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 + z^{-2})} = \frac{z^2}{z^2 + 1} \quad \text{system function}$$

The signal has same z-transform as system. The signal is the impulse response of the system

$$\begin{aligned} \text{zeros} &= \text{roots}(z^2) = 0, 0 \\ \text{poles} &= \text{roots}(z^2 + 1) = \pm j \end{aligned}$$

Poles: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \rightarrow \infty$

Zeros: values of z for input $x[n] = z^n$ where output $y[n] = H(z)z^n \rightarrow 0$

$$y[n] = h[n] = \cos\left(\frac{2\pi}{4}n\right)u[n] \quad \text{signal}$$

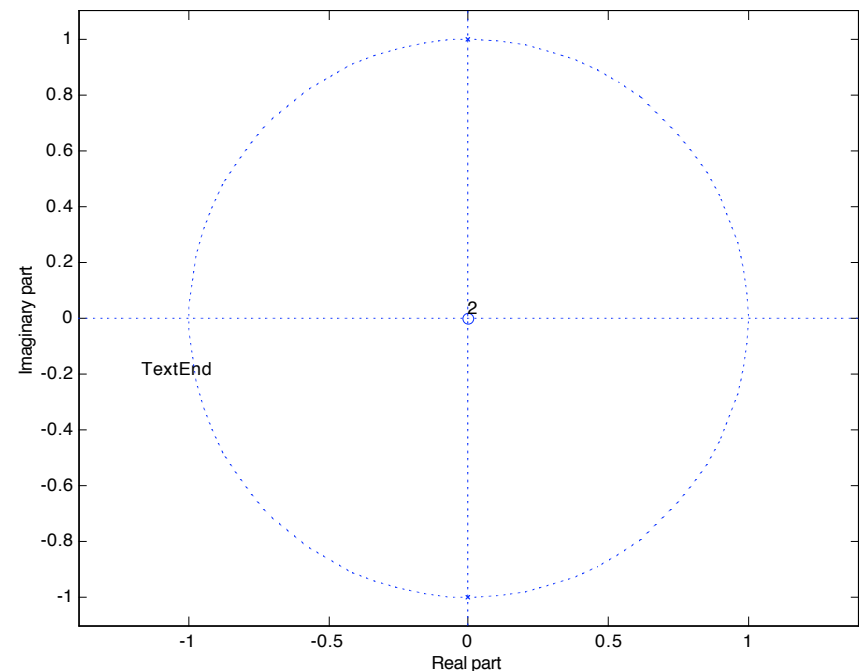
↓ z-transform

$$Y(z) = \frac{1}{(1 + z^{-2})} = \frac{z^2}{z^2 + 1}$$

Poles: z locations of $y[n] = z^n$

Zeros: related to the magnitude and phase of $y[n] = z^n$

The closer a zero is to a pole, the smaller the effect the pole.



DFT Convolution

ex.

$$x[n]=[1 \ -1 \ 1], \quad y[n]=[1 \ 2 \ 3]$$

$$z[n]=x[n]*y[n]= [1 \ 1 \ 2 \ -1 \ 3]$$

$$\begin{array}{rcccc} \mathbf{x[n]} & 1 & -1 & 1 & \\ \mathbf{y[n]} & 1 & 2 & 3 & \\ \hline & & 1 & 2 & 3 \\ & & & -1 & -2 & -3 \\ & & & & 1 & 2 & 3 \\ \hline \mathbf{z[n]} & 1 & 1 & 2 & -1 & 3 \end{array}$$

3pt DFT

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k/N)n} = \sum_{n=0}^2 x[n] e^{-j(2\pi k/3)n} \\ &= x[0] e^{-j(2\pi 0/3)k} + x[1] e^{-j(2\pi 1/3)k} + x[2] e^{-j(2\pi 2/3)k} \\ &= 1 - e^{-j(2\pi/3)k} + e^{-j(4\pi/3)k} \end{aligned}$$

$$X[0] = 1 - e^{-j(2\pi/3)0} + e^{-j(4\pi/3)0} = 1 - 1 + 1 = 1$$

$$X[1] = 1 - e^{-j(2\pi/3)^1} + e^{-j(4\pi/3)^1} = 1 + j1.7321$$

$$X[2] = 1 - e^{-j(2\pi/3)^2} + e^{-j(4\pi/3)^2} = 1 - 1.7321j$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

3pt IDFT $Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$

$$\begin{aligned} z[n] &= \frac{1}{N} \sum_{k=0}^{N-1} Z[k] e^{-j(2\pi k/N)n} = \frac{1}{3} \sum_{k=0}^2 Z[k] e^{-j(2\pi k/3)n} \\ &= \left(Z[0] e^{-j(2\pi 0/3)k} + Z[1] e^{-j(2\pi 1/3)k} + Z[2] e^{-j(2\pi 2/3)k} \right) / 3 \\ &= \left(6 + (-3 - j1.73) e^{-j(2\pi/3)k} + (-3 + j1.73) e^{-j(4\pi/3)k} \right) / 3 \end{aligned}$$

$$Y[0] = 1 + 2e^{-j(2\pi/3)0} + 3e^{-j(4\pi/3)0} = 1 + 2 + 3 = 6$$

$$Y[1] = 1 + 2e^{-j(2\pi/3)1} + 3e^{-j(4\pi/3)1} = -1.5 + j0.866$$

$$Y[2] = 1 + 2e^{-j(2\pi/3)2} + 3e^{-j(4\pi/3)2} = -1.5 - j0.866$$

$$Y[k] = [6 \quad -1.5 + j0.866 \quad -1.5 - j0.866]$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

3pt DFT $y[n]=[1 \ 2 \ 3]$

$$\begin{aligned}
 Y[k] &= \sum_{n=0}^{N-1} Y[n]e^{-j(2\pi k/N)n} = \sum_{n=0}^2 y[n]e^{-j(2\pi k/3)n} \\
 &= y[0]e^{-j(2\pi 0/3)k} + y[1]e^{-j(2\pi 1/3)k} + y[2]e^{-j(2\pi 2/3)k} \\
 &= 1 - 2e^{-j(2\pi/3)k} + 3e^{-j(4\pi/3)k}
 \end{aligned}$$

$$Y[0] = 1 + 2e^{-j(2\pi/3)0} + 3e^{-j(4\pi/3)0} = 1 + 2 + 3 = 6$$

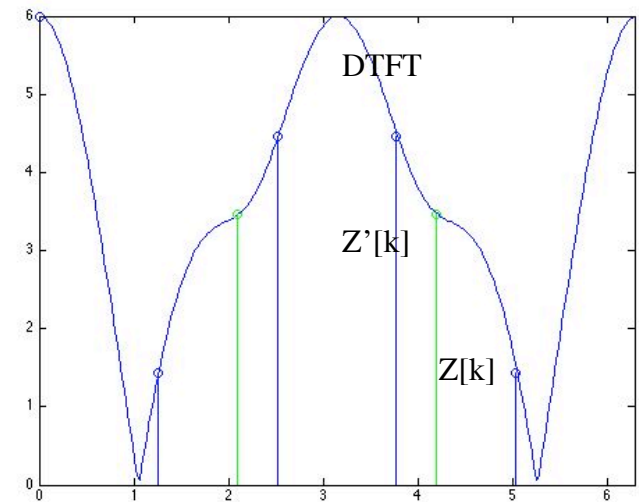
$$Y[1] = 1 + 2e^{-j(2\pi/3)1} + 3e^{-j(4\pi/3)1} = -1.5 + j0.866$$

$$Y[2] = 1 + 2e^{-j(2\pi/3)2} + 3e^{-j(4\pi/3)2} = -1.5 - j0.866$$

$$Y[k] = [6 \quad -1.5 + j0.866 \quad -1.5 - j0.866]$$

$$X[k] = [1 \quad 1 + j1.7321 \quad 1 - j1.7321]$$

$$Z[k] = X[k]Y[k] = [6 \quad -3 - j1.73 \quad -3 + j1.73]$$



$$z'[n]=[1 \ 1 \ 2 \ -1 \ 3]$$

$$Z'[k] = [6 \quad 1.43 + j0.139 \quad -1.93 + j4.03 \quad -1.93 - j4.03 \quad 1.43 - j0.139]$$

Infinite signals

$$\begin{aligned}x[n] &= -a^n u[-n-1] \Leftrightarrow X(z) = -\sum_{k=-\infty}^{-1} a^k z^{-k} = -\sum_{k=1}^{\infty} a^{-k} z^k \\x[n] &= 0 \quad n \geq 0 \\&\text{left sided} \\&= -\sum_{k=1}^{\infty} \left(\frac{1}{a} z\right)^k = 1 - \sum_{k=0}^{\infty} \left(\frac{1}{a} z\right)^k \\&= 1 - \frac{1}{1 - \left(\frac{1}{a} z\right)} = \frac{1 - \left(\frac{1}{a} z\right) - 1}{1 - \left(\frac{1}{a} z\right)} \\&= \frac{-\left(\frac{1}{a} z\right)}{1 - \left(\frac{1}{a} z\right)} = \frac{-1}{\left(\frac{1}{a} z\right)^{-1} - 1} = \frac{1}{1 - az^{-1}}\end{aligned}$$

$$\begin{aligned}X(z) &= \frac{1}{1 - az^{-1}} & \left|\frac{1}{a} z\right| < 1 & \text{region of convergence} \\& & \text{or } |z| < |a| & \end{aligned}$$