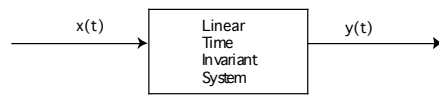


Linear Time Invariant Systems



Linearity

$$\begin{array}{ccc} \text{input} & & \text{output} \\ Ax_1(t) + Bx_2(t) & \longrightarrow & Ay_1(t) + By_2(t) \end{array}$$

scaling & superposition

Time invariance

$$x(t-\tau) \longrightarrow y(t-\tau)$$

Characteristic Functions

$$e^{st} \quad s = a + jb$$

complex exponentials

Complex Exponential Signals

$$s = a + jb$$

$$e^{st}$$

$$s = \sigma$$

$$e^{-\sigma t}$$

exponential decay

$$s = \pm j\omega$$

$$e^{\pm j\omega t}$$

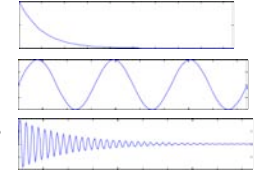
sinusoids

$$s = -\sigma \pm j\omega$$

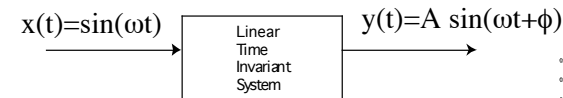
$$e^{-\sigma t} e^{\pm j\omega t}$$

exponential sinusoids

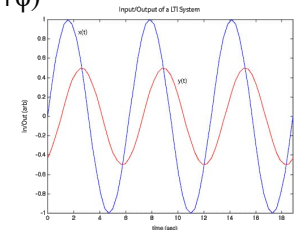
$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$



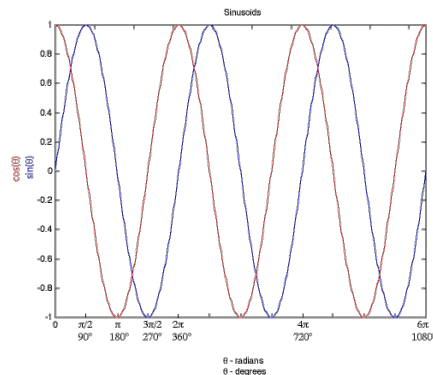
characteristic functions of LTI systems



output has same frequency as input
but is scaled and phase shifted



Sinusoids



θ (rad)	$y_1 = \sin(\theta)$	$y_2 = \cos(\theta)$
0	0	1
$\pi/6$	.5	0.866
$\pi/4$	0.707	0.707
$\pi/3$	0.866	0.5
$\pi/2$	1	0
π	0	-1
$3\pi/2$	-1	0
2π	0	1

Periodic

$$y(\theta) = y(\theta + 2\pi n)$$

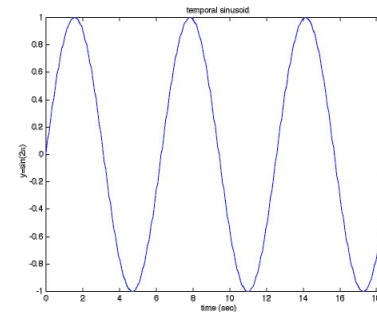
sine odd

$$\sin(-\theta) = -\sin(\theta)$$

cosine even

$$\cos(-\theta) = \cos(\theta)$$

Continuous sinusoids $\theta = \theta(t)$



$$y(t) = A \sin(\omega t + \phi)$$

$$y(t) = A \sin(2\pi f t + \phi)$$

Parameters:

A: amplitude

ϕ : phase (radians)

ω : radian frequency (radians/sec)

or

f: frequency (cycles/sec-Hz)

Relations:

$$\omega = 2\pi f$$

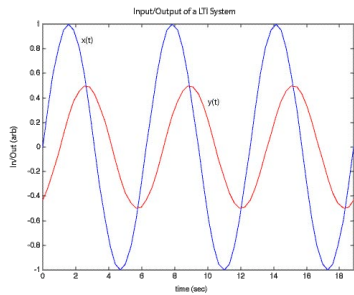
$$\text{rad/sec} = (2\pi \text{ rad/cycle}) * \text{cycle/sec}$$

$$T: \text{period (sec/cycle)} \quad y(t) = y(t+T)$$

$$T = 1/f = 2\pi/\omega$$

$$\text{sec/cycle} = 1/(\text{cycle/sec}) = (2\pi \text{ rad/cycle})/(\text{rad/sec})$$

Continuous sinusoids $\theta=\theta(t)$



$$y(t) = A \sin(\omega t + \phi)$$

$$y(t) = A \sin(2\pi f t + \phi)$$

Parameters:

A: amplitude

ϕ : phase (radians)

ω : radian frequency (radians/sec)

or

f: frequency (cycles/sec-Hz)

Phase shift

In: $x(t) = 1 \sin(t)$

Out: $y(t) = 0.5 \sin(t - \pi/3)$

$x(0) = 0$

$y(\pi/3) = 0$

Sampled Continuous Sinusoid

Continuous Sinusoid

$$y(t) = \sin(2\pi t)$$

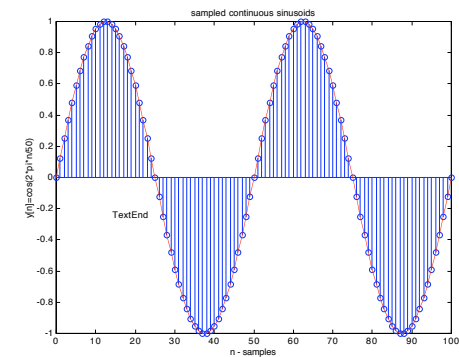
Sample rate: $T_s = \frac{1}{50}$ sec

$$t = nT_s$$

Discrete Sinusoid

$$y[n] = \sin(2\pi \cdot \frac{1}{50} \cdot n)$$

$$y[n] = \sin(\frac{\pi}{25} \cdot n)$$



n	y[n]
0	0
1	0.125
2	0.249
3	0.368
4	0.4818

Discrete sinusoids $\theta=\theta[n]$ $n=0, 1, 2, \dots$

$$y[n] = A \sin(\omega n + \phi)$$

$$y[n] = A \sin(2\pi f n + \phi)$$

A: amplitude

ϕ : phase (radians)

ω : radian frequency (radians/sample)

f: frequency (cycles/sample)

Relations:

$$\omega = 2\pi f \text{ rad/sample} = (2\pi \text{ rad/cycle}) * \text{cycle/sample}$$

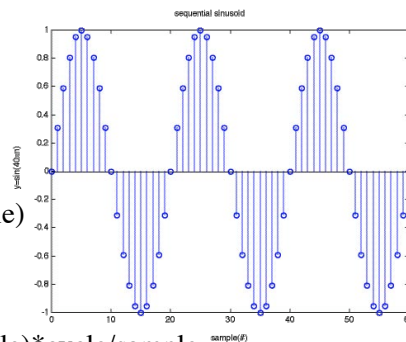
N: period (samples/repeating cycle [integer])

Smallest integer N such that $y[n] = y[n+N]$

Find an integer k so $N = k/f$ is also an integer

$$N \neq \frac{1}{f}$$

$f = k/N$ (f: rational number $\rightarrow k/N$ is ratio of integers)



Period of discrete sinusoids

$$\text{ex: } y[n] = \cos(2\pi(3/16)n)$$

What is the period N?

$$y[n] = A \cos(2\pi f n + \phi)$$

frequency: $f = 3/16$ cycles/sample

N: period

(samples/repeating cycle [integer])

Smallest integer N such that $y[n] = y[n+N]$

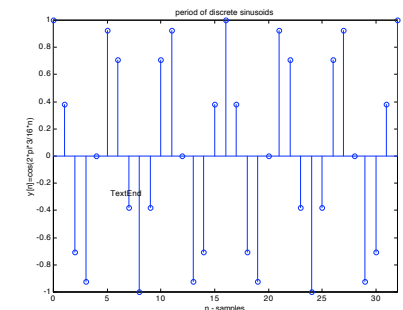
Find an integer k so $N = k/f$ is also an integer

$$f = 3/16$$

$$\text{let } k = 3$$

$$N = (3) * 16/3 = 16$$

$f = k/N$ (rational number $\rightarrow k/N$ is ratio of integers)



Period of discrete sinusoids: ex2.

$$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$$

$$f = \frac{3}{50} \frac{\text{cycles}}{\text{sample}}$$

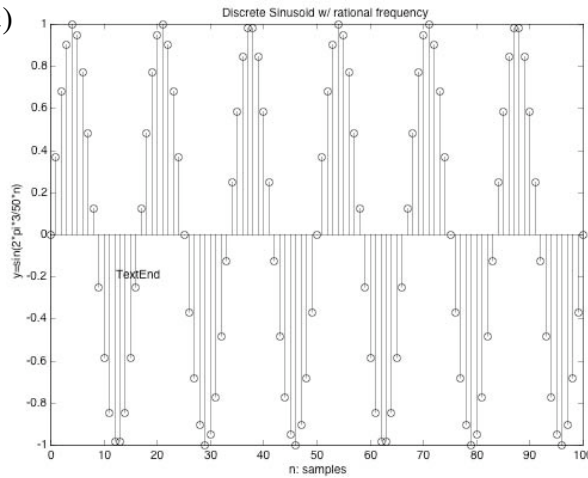
$$y[n] = y[n + N]$$

N=?? samples

N: integer

$$N \neq \frac{1}{f}$$

50/3 $\neq$ integer



Period of discrete sinusoids: ex2.

discrete function

$$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$$

$$\text{frequency: } f = \frac{3}{50} \frac{\text{cycles}}{\text{sec}}$$

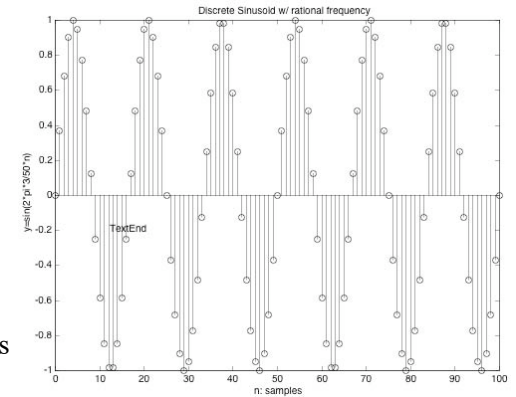
period: N=?? samples

$$y[n] = y[n + N]$$

$$f \cdot N = k \quad N, k: \text{integers}$$

$$\frac{3}{50} \cdot N = k$$

$$\frac{N}{k} = \frac{50}{3} \frac{\text{samples}}{\text{cycle}} \quad \begin{array}{l} \text{ratio of integers} \\ \text{rational number} \end{array} \quad \begin{array}{l} \text{periodic} \\ N=50 \text{ samples, } k=3 \text{ cycles} \end{array}$$



Aperiodic discrete sinusoids

continuous function

$$y(t) = \sin(2\pi \cdot \sqrt{2} \cdot t)$$

$$T = \frac{1}{\sqrt{2}} \text{ sec} \quad \text{periodic}$$

sample

$$t = nT_s \quad T_s = \frac{1}{25} \text{ sec}$$

discrete function

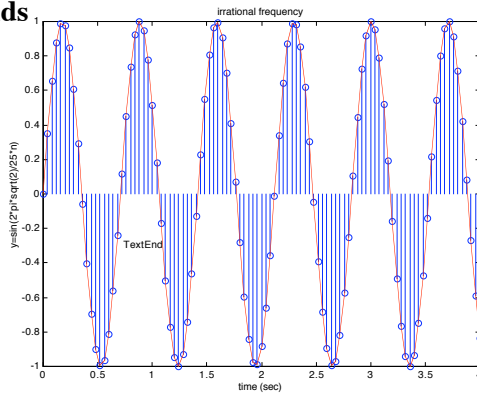
$$y[n] = \sin(2\pi \cdot \frac{\sqrt{2}}{25} \cdot n)$$

period?

$$y[n] = y[n + N]$$

N=?? samples

(integer)



$$f = \frac{\sqrt{2}}{25}$$

$$f \cdot N = k \quad N, k: \text{integers}$$

$$\frac{N}{k} = \frac{25\sqrt{2}}{2} \quad \begin{array}{l} \text{not a ratio of integers} \\ \text{irrational number} \end{array}$$

sampled discrete sinusoid aperiodic

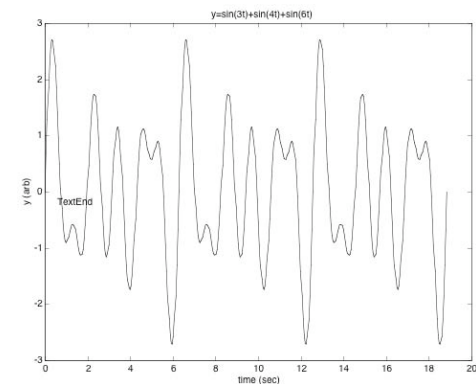
Periodicity

arbitrary continuous signal

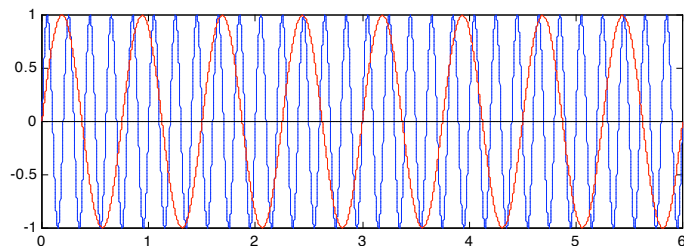
T: period (sec/cycle)

$$y(t+T) = y(t)$$

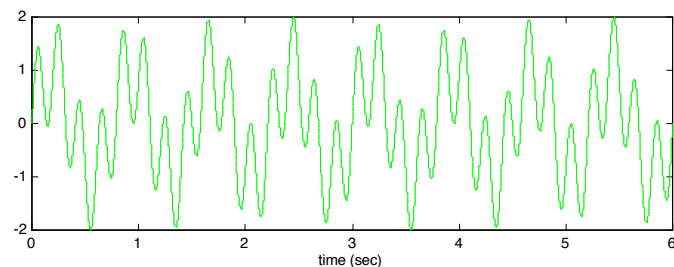
After what interval does the signal repeat itself?



Ex: Period of sum of sinusoids



T1=0.2 seconds, T2=0.75 seconds



Tsum=? seconds

Instantaneous frequency

$$y(\theta) = \sin(\theta) \quad \text{time varying argument}$$

$$\theta = \theta(t)$$

instantaneous frequency

$$\omega = d\theta/dt$$

sinusoid constant frequency

$$y(t) = A \sin(\omega t + \phi)$$

$$\theta = \omega t + \phi$$

$$d\theta/dt = \omega$$

chirp linearly swept frequency

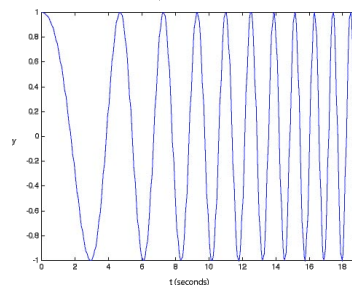
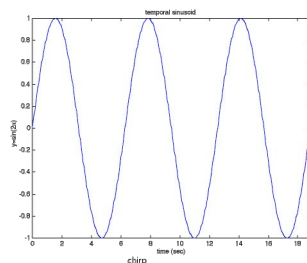
$$\omega = d\theta/dt$$

$$\omega = ((\omega_1 - \omega_0)/T)t + \omega_0$$

integrate

$$\theta = (\omega_1 - \omega_0)/(2T) t^2 + \omega_0 t + C$$

$$y_{\text{chirp}}(t) = A \sin((\omega_1 - \omega_0)/(2T) t^2 + \omega_0 t + \phi)$$



Least common multiple

seconds to complete cycles

$$T1 = 1/5 \text{ seconds}$$

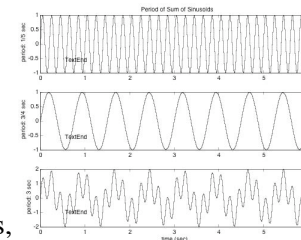
$$1/5s, 2/5s, 3/5s \dots$$

4/20s, 8/20s, 12/20s,
16/20s, 20/20s, 24/20s,
28/20s, 32/20s, 36/20s,
40/20s, 44/20s, 48/20s,
52/20s, 56/20s, 60/20s

15 cycles

$$T_{\text{sum}} = 15 * T1 = 15/5 = 3 \text{ seconds} \quad T_{\text{sum}} = 4 * T2 = 3/4 * 4 = 3 \text{ seconds}$$

$$T_{\text{sum}} = 3 \text{ seconds}$$



seconds to complete cycles

$$T2 = 3/4 \text{ seconds}$$

$$3/4s, 6/4s, \dots$$

15/20s, 30/20s,
45/20s, 60/20s

4 cycles

$$1/5 * k = 3/4 * 1$$

$$k/l = 15/4$$

rational number

Representations of a sinusoid

$$y(t) = A \cos(\omega t + \phi) \quad \text{trig function}$$

$$y(t) = A e^{j\phi} \frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad \text{complex conjugates} \quad e^{j(\theta)} = e^{j(\omega t + \phi)}$$

$$= e^{j\phi} e^{j\omega t}$$

$$y(t) = \text{Re}\{A e^{j\phi} e^{j(\omega t)}\} \quad \text{real part of complex exponential}$$

$$X = A e^{j\phi} \quad \text{complex amplitude (constant)}$$

$$y(t) = \text{Re}\{X e^{j(\omega t)}\} \quad \text{rotating phasor}$$

Euler's relations

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Exponentials

Why use complex exponentials?

Trigonometric manipulations -> algebraic operations on exponents

Trigonometric identities

$$\cos(x)\cos(y) = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Properties of exponentials

$$re^xe^y = re^{x+y}$$

$$(re^x)^n = r^n e^{nx}$$

$$\sqrt[n]{x} = x^{1/n}$$

$$\frac{1}{x} = x^{-1}$$

Vector representation (graphical)

Representations of a sinusoid

$$y(t) = A \cos(\omega t + \phi)$$

trig function

$$y(t) = A e^{j\phi} \frac{[e^{j\omega t} + e^{-j\omega t}]}{2}$$

complex conjugates

$$e^{j(\theta)} = e^{j(\omega t + \phi)} \\ = e^{j\phi} e^{j\omega t}$$

$$y(t) = \text{Re}\{A e^{j\phi} e^{j(\omega t)}\}$$

real part of
complex exponential

$$X = A e^{j\phi} \quad \text{complex amplitude (constant)}$$

$$y(t) = \text{Re}\{X e^{j(\omega t)}\} \quad \text{rotating phasor}$$

Euler's relations

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Exponentials

Amplitude modulation (multiply two sinusoids of different frequencies)

$$A \cos(\omega_1 t) B \cos(\omega_2 t + \phi) = C(\cos(\omega_3 t + \phi_2) + \cos(\omega_4 t + \phi_2))$$

$$\begin{aligned} & A \cdot \cos(\omega_1) \cdot (B \cdot \cos(\omega_2 + \phi_2)) \\ & A \cdot B \cdot \cos(\omega_1) \cdot ((\cos(\omega_2) \cdot \cos(\phi_2) - \sin(\omega_2) \cdot \sin(\phi_2))) \\ & A \cdot B \cdot \cos(\omega_1) \cdot \cos(\phi_2) - A \cdot B \cdot \cos(\omega_1) \cdot \sin(\omega_2) \cdot \sin(\phi_2) \\ & A \cdot B \cdot \frac{\cos(\omega_2 + \omega_1) + \cos(\omega_2 - \omega_1)}{2} \cdot \cos(\phi_2) - A \cdot B \cdot \frac{\sin(\omega_2 + \omega_1) + \sin(\omega_2 - \omega_1)}{2} \cdot \sin(\phi_2) \\ & A \cdot B \cdot \frac{\cos(\omega_s) + \cos(\omega_d)}{2} \cdot \cos(\phi_2) - A \cdot B \cdot \frac{\sin(\omega_s) + \sin(\omega_d)}{2} \cdot \sin(\phi_2) \\ & \frac{1}{2} \cdot A \cdot B \cdot (\cos(\phi_2) \cdot \cos(\omega_s) + \cos(\phi_2) \cdot \cos(\omega_d) - \sin(\phi_2) \cdot \sin(\omega_s) - \sin(\phi_2) \cdot \sin(\omega_d)) \\ & \frac{1}{2} \cdot A \cdot B \cdot (\cos(\omega_s + \phi_2) + \sin(\omega_d + \phi_2)) \end{aligned}$$

*sum formula
for sin & cos
trig id
*product formula
for sin & cos
trig id

*sum formula
for sin & cos
trig id

or

$$\begin{aligned} & A \cdot \cos(\omega_1) \cdot (B \cdot \cos(\omega_2 + \phi_2)) \\ & A \cdot \left[\frac{e^{j\omega_1} + e^{-j\omega_1}}{2} \right] \cdot \left[B \cdot \left[\frac{e^{j(\omega_2 + \phi)} + e^{-j(\omega_2 + \phi)}}{2} \right] \right] \\ & \frac{1}{4} \cdot A \cdot B \cdot [\exp[j \cdot (\phi + \omega_2 + \omega_1)] + \exp[-j \cdot (\phi + \omega_2 + \omega_1)] + \exp[-j \cdot (\phi + \omega_2 - \omega_1)] + \exp[j \cdot (\phi + \omega_2 - \omega_1)]] \\ & \frac{1}{4} \cdot A \cdot B \cdot (2 \cdot \cos(\phi + \omega_2 + \omega_1) + 2 \cdot \cos(\phi + \omega_2 - \omega_1)) \\ & \frac{1}{2} \cdot A \cdot B \cdot (\cos(\phi + \omega_2 + \omega_1) + \cos(\phi + \omega_2 - \omega_1)) \end{aligned}$$

cos = complex conj

mult. exponentials

cos = complex conj

complex numbers

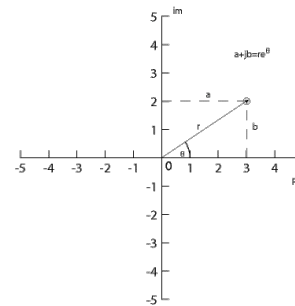
cartesian

$$s = a + jb$$

$$j = \sqrt{-1}$$

polar

$$s = r e^{j\theta}$$



conversion

$$a = r \cos(\theta)$$

$$b = r \sin(\theta)$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$e^{j0} = 1$$

$$e^{j\pi/2} = j$$

$$e^{j\pi} = -1$$

quadrants!

conjugate

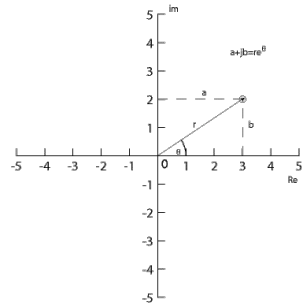
$$s^* = a - jb = r e^{-j\theta}$$

complex numbers

cartesian $s = a + jb$

$$j = \sqrt{-1}$$

polar $s = re^{j\theta}$



conversion

$$a = r \cos(\theta)$$

$$b = r \sin(\theta)$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = a \tan\left(\frac{b}{a}\right)$$

conjugate

$$s^* = a - jb = re^{-j\theta}$$

remember:

$$e^{j0} = 1$$

$$e^{j\pi/2} = j$$

$$e^{j\pi} = -1$$

ex.

$$1j = (e^{j0})j$$

$$= e^{j(j)(0)}$$

$$= e^{-1(0)}$$

$$= e^0 = 1$$

ex. $\cos(j)$?

$$1/j?$$

complex numbers

cartesian

$$s_1 = 3 + j2$$

$$= \sqrt{3^2 + 2^2} \cdot e^{j \cdot \text{atan}\left[\frac{2}{3}\right]}$$

$$= \sqrt{13} \cdot e^{j0.588}$$

$$s_2 = -2 + j1$$

$$= \sqrt{(-2)^2 + 1^2} \cdot e^{j \left[\text{atan}\left[\frac{1}{-2}\right] + \pi \right]}$$

$$= \sqrt{5} \cdot e^{j2.678}$$

polar

$$s_1 = \sqrt{13} \cdot e^{j0.588}$$

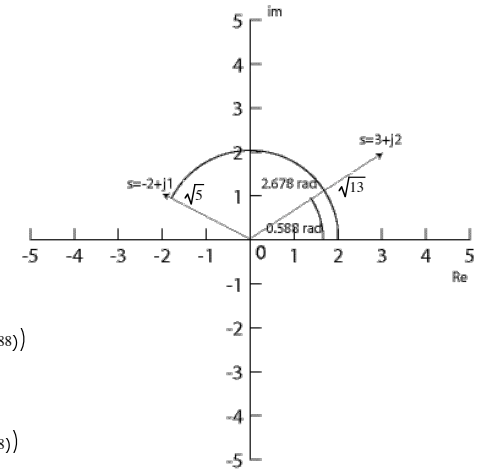
$$= \sqrt{13} \cdot \cos(0.588) + j \cdot (\sqrt{13} \cdot \sin(0.588))$$

$$= 3 + j2$$

$$s_2 = \sqrt{5} \cdot e^{j2.678}$$

$$= \sqrt{5} \cdot \cos(2.678) + j \cdot (\sqrt{5} \cdot \sin(2.678))$$

$$= -2 + j1$$



Addition

cartesian

$$s_1 = a_1 + jb_1$$

$$s_2 = a_2 + jb_2$$

$$s_1 + s_2 = a_1 + jb_1 + a_2 + jb_2$$

$$= (a_1 + a_2) + j(b_1 + b_2)$$

polar

$$s_1 = r_1 e^{j\theta_1}$$

$$s_2 = r_2 e^{j\theta_2}$$

convert to cartesian

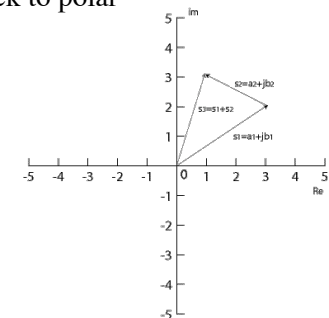
add

convert back to polar

vector

place tail of s_2 at
head of s_1

connect origin to s_2



Addition - example

cartesian

$$\begin{aligned}s_1 &= 3 + j2 \\ s_2 &= -2 + j1\end{aligned}$$

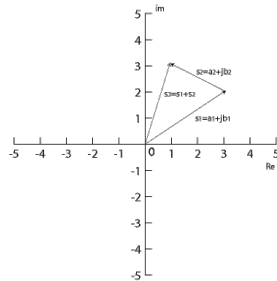
$$\begin{aligned}s_1 + s_2 &= a_1 + a_2 + j(b_1 + b_2) \\ &= 3 - 2 + j(2 + 1) \\ &= 1 + j3\end{aligned}$$

polar

$$\begin{aligned}s_1 &= \sqrt{13} e^{j0.588} \\ s_2 &= \sqrt{5} e^{j2.678}\end{aligned}$$

$$\begin{aligned}s_1 &= \sqrt{13} \cos(0.588) + j\sqrt{13} \sin(0.588) = 3 + j2 \\ s_2 &= \sqrt{5} \cos(2.678) + j\sqrt{5} \sin(2.678) = -2 + j1\end{aligned}$$

vector



$$\begin{aligned}s_1 + s_2 &= 1 + j3 \\ &= \sqrt{1^2 + 3^2} \cdot e^{j \cdot \text{atan}(3)} \\ &= \sqrt{10} \cdot e^{j \cdot 1.249}\end{aligned}$$

Subtraction

cartesian

$$\begin{aligned}s_1 &= a_1 + jb_1 \\ s_2 &= a_2 + jb_2\end{aligned}$$

$$\begin{aligned}s_1 - s_2 &= a_1 + jb_1 - (a_2 + jb_2) \\ &= (a_1 - a_2) + j(b_1 - b_2)\end{aligned}$$

polar

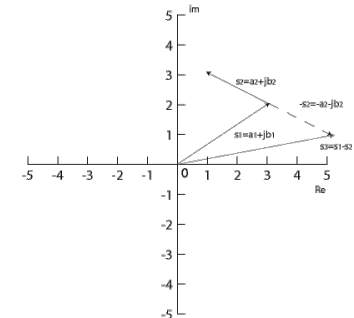
$$\begin{aligned}s_1 &= r_1 e^{j\theta_1} \\ s_2 &= r_2 e^{j\theta_2}\end{aligned}$$

convert to cartesian
add
convert back to polar

vector

rotate s_2 180° ($-s_2$)
place tail of $-s_2$ at
head of s_1

connect origin to s_2



Subtraction - example

cartesian

$$\begin{aligned}s_1 &= 3 + j2 \\ s_2 &= -2 + j1\end{aligned}$$

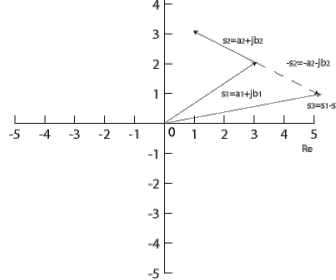
$$\begin{aligned}s_1 - s_2 &= (a_1 - a_2) + j(b_1 - b_2) \\ &= (3 - (-2)) + j(2 - 1) \\ &= 5 + j1\end{aligned}$$

polar

$$\begin{aligned}s_1 &= \sqrt{13} e^{j0.588} \\ s_2 &= \sqrt{5} e^{j2.678}\end{aligned}$$

$$\begin{aligned}s_1 &= \sqrt{13} \cos(0.588) + j\sqrt{13} \sin(0.588) = 3 + j2 \\ s_2 &= \sqrt{5} \cos(2.678) + j\sqrt{5} \sin(2.678) = -2 + j1\end{aligned}$$

vector



$$\begin{aligned}s_1 - s_2 &= 5 + j1 \\ &= \sqrt{5^2 + 1^2} \cdot e^{j \cdot \text{atan}\left[\frac{1}{5}\right]} \\ &= \sqrt{26} \cdot e^{j \cdot 0.197}\end{aligned}$$

Multiplication

cartesian

$$\begin{aligned}s_1 &= a_1 + jb_1 \\ s_2 &= a_2 + jb_2\end{aligned}$$

$$\begin{aligned}s_1 s_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= a_1 a_2 + ja_1 b_2 + ja_2 b_1 + j^2 b_1 b_2 \\ &= a_1 a_2 + j^2 b_1 b_2 + j(a_1 b_2 + a_2 b_1) \\ &= a_1 a_2 - b_1 b_2 + j(a_1 b_2 + a_2 b_1)\end{aligned}$$

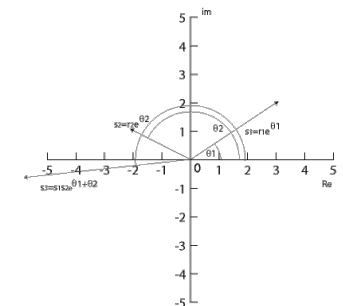
polar

$$\begin{aligned}s_1 &= r_1 e^{j\theta_1} \\ s_2 &= r_2 e^{j\theta_2}\end{aligned}$$

$$\begin{aligned}s_1 s_2 &= r_1 e^{j\theta_1} r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j\theta_1} e^{j\theta_2} \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)}\end{aligned}$$

vector

magnitudes multiply
angles add



$$x^a x^b = x^{a+b}$$

Multiplication - example

cartesian

$$s_1 = 3 + j2$$

$$s_2 = -2 + j1$$

$$s_1 s_2 = a_1 a_2 - b_1 b_2 + j(a_1 b_2 + a_2 b_1)$$

$$= 3(-2) - 2(1) + j(3(1) + (-2)(2))$$

$$= -6 - 2 + j(3 - 4)$$

$$= -8 - j1$$

polar

$$s_1 = \sqrt{13} e^{j0.588}$$

$$s_2 = \sqrt{5} e^{j2.678}$$

$$s_1 s_2 = r_1 r_2 e^{j\theta_1 + \theta_2}$$

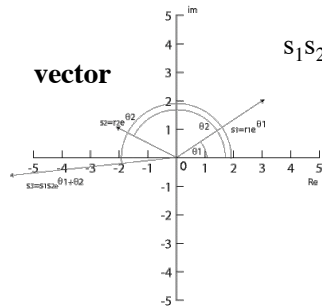
$$= \sqrt{13} \sqrt{5} e^{j(0.588 + 2.678)}$$

$$= \sqrt{65} e^{j3.266}$$

$$s_1 s_2 = \sqrt{65} \cos(3.266) + j \sqrt{65} \sin(3.266)$$

$$= -8 - j1$$

vector



Division

cartesian

$$s_1 = a_1 + j b_1$$

$$s_2 = a_2 + j b_2$$

$$s_1 / s_2 = (a_1 + j b_1) / (a_2 + j b_2)$$

$$= (a_1 + j b_1) (a_2 - j b_2) / (a_2 + j b_2) (a_2 - j b_2)$$

$$= (a_1 + j b_1) (a_2 - j b_2) / (a_2^2 + b_2^2)$$

$$= s_1 s_2^* / |s_2|^2$$

polar

$$s_1 = r_1 e^{j\theta_1}$$

$$s_2 = r_2 e^{j\theta_2}$$

$$s_1 / s_2 = r_1 e^{j\theta_1} / r_2 e^{j\theta_2}$$

$$= r_1 e^{j\theta_1} (r_2 e^{j\theta_2})^{-1}$$

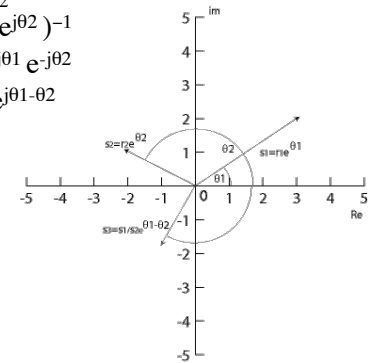
$$= (r_1 / r_2) e^{j\theta_1} e^{-j\theta_2}$$

$$= (r_1 / r_2) e^{j\theta_1 - \theta_2}$$

vector

divide magnitudes
angles subtract

$$x^a / x^b = x^{a-b}$$



Division - example

cartesian

$$s_1 = 3 + j2$$

$$s_2 = -2 + j1$$

$$s_1 / s_2 = s_1 s_2^* / |s_2|^2$$

$$= (a_1 + j b_1) (a_2 - j b_2) / (a_2^2 + b_2^2)$$

$$= (3 + j2) (-2 - j1) / (2^2 + 1^2)$$

$$= (3(-2) - j3(1) + j2(-2) + 2(1)) / (5)$$

$$= (-6 - j3 - j4 + 2) / (5)$$

$$= (-4 - j7) / (5)$$

polar

$$s_1 = \sqrt{13} e^{j0.588}$$

$$s_2 = \sqrt{5} e^{j2.678}$$

$$s_1 / s_2 = (r_1 / r_2) e^{j\theta_1 - \theta_2}$$

$$= (\sqrt{13} / \sqrt{5}) e^{j0.588 - 2.678}$$

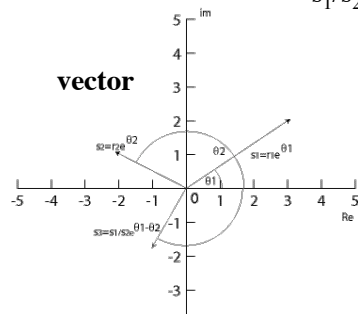
$$= (\sqrt{13} / \sqrt{5}) e^{j0.588 - 2.678}$$

$$= \sqrt{\frac{13}{5}} e^{-j2.09}$$

$$= \sqrt{\frac{13}{5}} \cos(-2.09) + j \sqrt{\frac{13}{5}} \sin(-2.09)$$

$$= -0.8 - j1.4$$

vector



Powers

cartesian

$$s = a + j b$$

$$s^n = (a + j b)^n$$

$$= \sum_{k=0}^n \binom{n}{k} a^{n-k} (j b)^k$$

Binomial expansion

$$\text{where } \binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-(k-1))}{k!}$$

polar

$$s = r e^{j\theta}$$

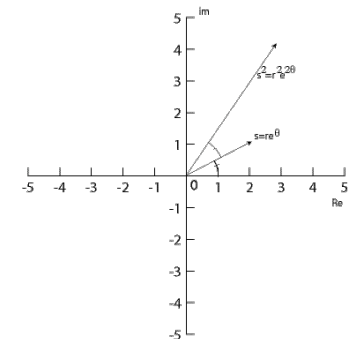
$$s^n = (r e^{j\theta})^n$$

$$s^n = r^n e^{j n \theta}$$

vector

magnitude raised to power n
angle multiplied by n

$$(x^a)^n = x^{a n}$$



Powers - example $s^2=(2+j1)^2$

cartesian

$$\begin{aligned} s &= 2+j1 \\ &= \sqrt{2^2 + 1^2} \cdot e^{j \cdot \text{atan}\left[\frac{1}{2}\right]} \\ &= \sqrt{5} \cdot e^{j \cdot 0.464} \end{aligned}$$

polar

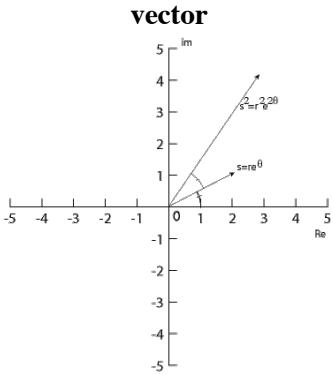
$$\begin{aligned} s &= r e^{j\theta} \\ &= \sqrt{5} \cdot e^{j \cdot 0.464} \end{aligned}$$

$$s^n = r^n e^{jn\theta}$$

$$s^2 = r^2 e^{j2\theta}$$

$$s^2 = \sqrt{5}^2 e^{j2(0.464)}$$

$$\begin{aligned} s^2 &= 5 e^{j0.927} \\ &= 5 \cos(0.927) + j5 \sin(0.927) \\ &= 3 + j4 \end{aligned}$$



Roots

cartesian

$$s = a + jb$$

$$s^{1/n} = (a + jb)^{1/n} \quad ???$$

$$= a^{1/n} \left(1 + j \frac{b}{a} \right)^{1/n} = 1 + j \frac{1}{n} \frac{b}{a} + \left(\frac{1/n}{2} \right) \left(j \frac{b}{a} \right)^2 + \left(\frac{1/n}{3} \right) \left(j \frac{b}{a} \right)^3 + \dots$$

polar

$$s = r e^{j\theta}$$

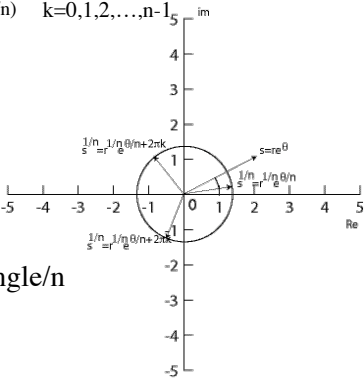
$$s^{1/n} = (r e^{j\theta})^{1/n}$$

$$s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)} \quad k=0, 1, 2, \dots, n-1$$

vector

nth positive root of magnitude
circle evenly divided by n starting at angle/n

$$\sqrt[n]{x} = x^{1/n}$$



Roots - example

cartesian

$$s = 2 + j1$$

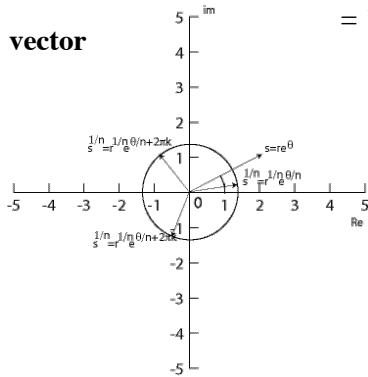
$$s^{1/3} = (a + jb)^{1/3}$$

$$s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)} \quad k=0, 1, 2, \dots, n-1$$

$$s^{1/3} = \sqrt[3]{5}^{1/3} e^{j(0.464/3 + 2\pi k/3)} \quad k=0, 1, 2$$

$$\begin{aligned} s^{1/3} &= 1.308 e^{j(0.464/3)} \\ &= 1.308 e^{j(0.464/3 + 2\pi 1/3)} \\ &= 1.308 e^{j(0.464/3 + 2\pi 2/3)} \end{aligned}$$

$$\begin{aligned} s^{1/3} &= 1.308 e^{j0.155} = 1.292 + j0.202 \\ &= 1.308 e^{j2.249} = -0.821 + j1.019 \\ &= 1.308 e^{j4.343} = -0.472 - j1.22 \end{aligned}$$



Complex Conversions

$$\begin{array}{ccc} \text{cartesian} & \longrightarrow & \text{polar} \\ s = a + jb & & s = \sqrt{a^2 + b^2} e^{j \cdot a \tan(b/a)} \end{array} \quad \begin{array}{ccc} \text{polar} & \longrightarrow & \text{cartesian} \\ s = r e^{j\theta} & & s = r \cos \theta + j r \sin \theta \end{array}$$

Complex Arithmetic		
Addition	cartesian	$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$
Subtraction	cartesian	$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$
Multiplication	polar	$r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
Division	polar	$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
Powers	polar	$(r e^{j\theta})^n = r^n e^{jn\theta}$
Roots	polar	$\begin{aligned} z^n &= s = r e^{j\theta} \\ z &= s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)} \quad k=1, 2, \dots, n-1 \end{aligned}$