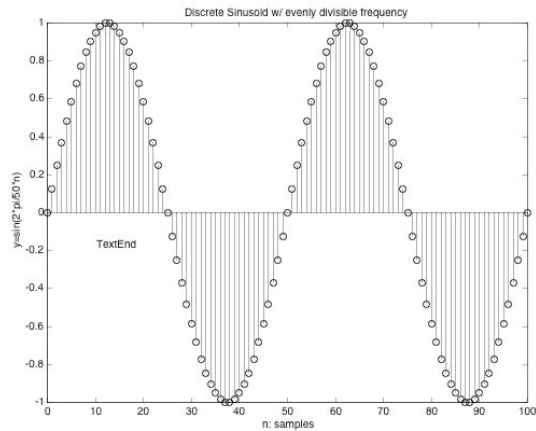


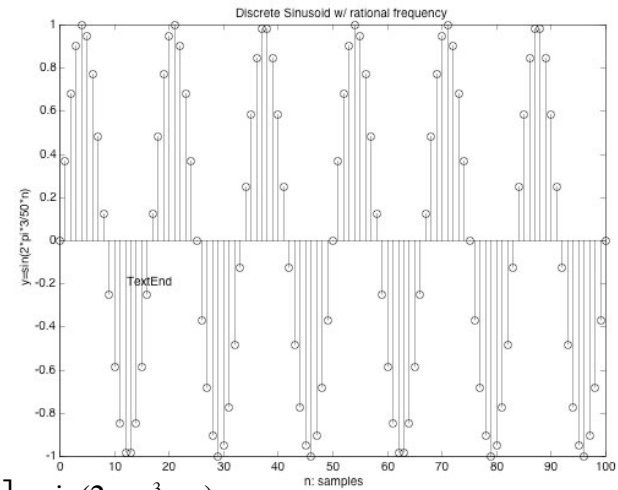
Period of a Discrete Sinusoid



$$y[n] = \sin(2\pi \cdot \frac{1}{50} \cdot n) \quad T=50 \text{ samples}$$

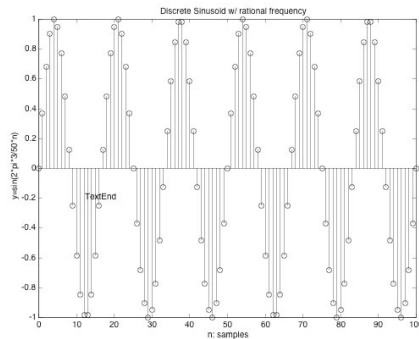
$$y[n] = y[n + 50]$$

$$\sin(0) = \sin(2\pi)$$



$$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$$

$$y[n] = y[n + T] \quad T=?? \text{ samples [integer]} \quad 50/3 \neq \text{integer}$$



$$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$$

$$y[n] = y[n + T] \quad T=?? \text{ samples}$$

$$\sin(0) = \sin(2\pi k) \quad k=1,2,\dots$$

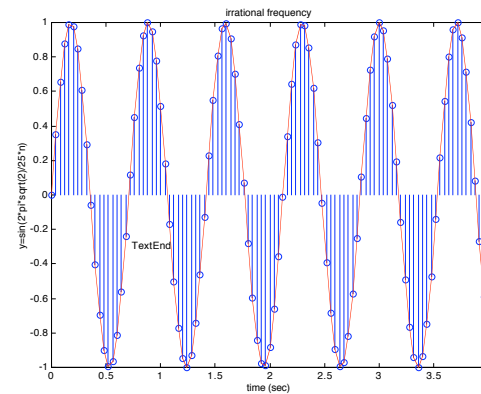
periodic

$$T=n=50 \text{ samples, } k=3 \text{ cycles}$$

$$2\pi \cdot \frac{3}{50} \cdot n = 2\pi k$$

$$\frac{n}{k} = \frac{50}{3} \frac{\text{samples}}{\text{cycle}} \quad \text{Ratio of integers}$$

rational number



$$T_s=1/25 \text{ sec}$$

$$y[n] = \sin(2\pi \cdot \frac{\sqrt{2}}{25} \cdot n)$$

$$y[n] = y[n + T] \quad T=?? \text{ samples}$$

$$\sin(0) = \sin(2\pi k) \quad k=1,2,\dots$$

Equiv. discrete sinusoid not periodic

$$y(t) = \sin(2\pi \cdot \sqrt{2} \cdot t) \quad T = \frac{1}{\sqrt{2}} \text{ sec}$$

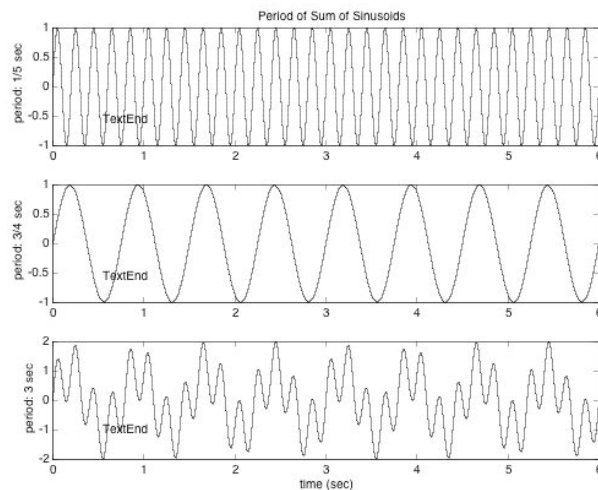
continuous function
periodic

$$2\pi \cdot \frac{\sqrt{2}}{25} \cdot n = 2\pi k$$

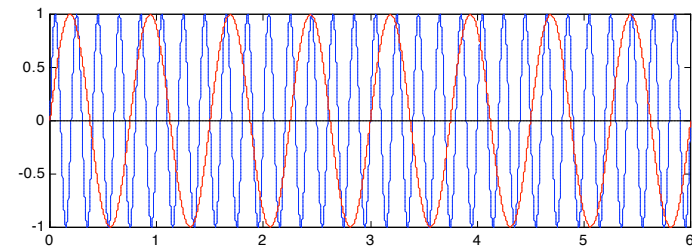
$$\frac{n}{k} = \frac{25\sqrt{2}}{2}$$

irrational number

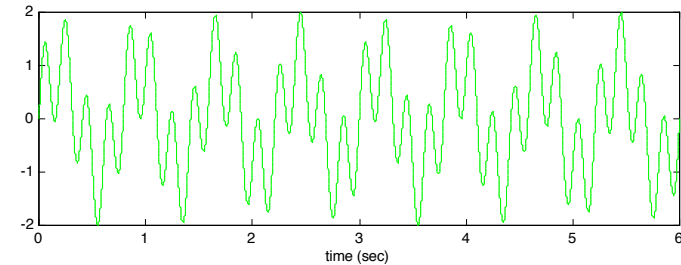
Period of Sum of Sinusoids



$$y(t) = y(t + T)$$



T1=0.2 seconds, T2=.75 seconds



Tsum=3 seconds

Least common multiple

seconds to complete cycles

T1=1/5 seconds

1/5s, 2/5s, 3/5s ...

4/20s, 8/20s, 12/20s,
16/20s, 20/20s, 24/20s,
28/20s, 32/20s, 36/20s,
40/20s, 44/20s, 48/20s,
52/20s, 56/20s, 60/20s

15 cycles

$$1/5 * k = 3/4 * 1$$

$$k/l = 15/4$$

rational number

seconds to complete cycles

T2=3/4 seconds

3/4s, 6/4s, ...

15/20s, 30/20s,
45/20s, 60/20s

4 cycles

Tsum=3 seconds

$$T_{\text{sum}} = 15 * T_1 = 15/5 = 3 \text{ seconds} \quad T_{\text{sum}} = 4 * T_2 = 3/4 * 4 = 3 \text{ seconds}$$

Complex Conversions

$$\begin{array}{ccc} \text{cartesian} & \longrightarrow & \text{polar} \\ s = a + jb & & s = \sqrt{a^2 + b^2} e^{j \cdot a \tan(b/a)} \end{array} \quad \begin{array}{ccc} \text{polar} & \longrightarrow & \text{cartesian} \\ s = r e^{j\theta} & & s = r \cos \theta + j r \sin \theta \end{array}$$

Complex Arithmetic

Addition	cartesian	$(a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)$
Subtraction	cartesian	$(a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$
Multiplication	polar	$r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
Division	polar	$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
Powers	polar	$(r e^{j\theta})^n = r^n e^{jn\theta}$
Roots	polar	$\begin{aligned} z^n &= S = r e^{j\theta} \\ z &= S^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)} \end{aligned} \quad k = 1, 2, \dots, n-1$

Complex Conversions

$$\begin{array}{ccc} \text{cartesian} & \longrightarrow & \text{polar} \\ 3 + j4 = \sqrt{3^2 + 4^2} e^{j \cdot \tan^{-1}(\frac{4}{3})} = 5e^{j \cdot 0.927} & & \text{polar} \longrightarrow \text{cartesian} \\ 2e^{j \cdot \frac{\pi}{3}} = 2 \cos \frac{\pi}{3} + j2 \sin \frac{\pi}{3} = 1 + j\sqrt{3} & & \end{array}$$

Complex Arithmetic		
Addition	cartesian	$(1 + j2) + (3 + j4) = (4 + j6)$
Subtraction	cartesian	$(1 + j2) - (3 + j4) = (-2 - j2)$
Multiplication	polar	$5e^{j \cdot \frac{\pi}{3}} \cdot 6e^{j \cdot \frac{\pi}{4}} = 5 \cdot 6e^{j(\frac{\pi}{3} + \frac{\pi}{4})} = 30e^{j \cdot \frac{7\pi}{12}}$
Division	polar	$10e^{j \cdot \frac{\pi}{2}} \div 5e^{j \cdot \frac{\pi}{4}} = (\frac{10}{5})e^{j(\frac{\pi}{2} - \frac{\pi}{4})} = 2e^{j \cdot \frac{\pi}{4}}$
Powers	polar	$(3e^{j \cdot \frac{\pi}{4}})^3 = 3^3 \cdot e^{j(\frac{3\pi}{4})} = 27e^{j(\frac{3\pi}{4})}$
Roots	polar	$z^3 = 64 = 64e^{j0}$ $z = 64^{1/3} e^{j(0/3 + 2\pi k/3)} = 4e^{j(2\pi k/3)}$

Representations of Sinusoids

$$\begin{aligned} A \cos(2\pi k f_0 t + \phi_k) &= \operatorname{Re}\{Ae^{j2\pi f t + \phi}\} \\ &= \operatorname{Re}\{Ae^{j\phi} e^{j2\pi f t}\} \\ &= \operatorname{Re}\{Xe^{j2\pi f t}\} \end{aligned} \quad Ae^{j\phi} \cdot \left(\frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2}\right) = X \cdot \left(\frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2}\right)$$

Sum multiple cosines same frequency

$$\begin{aligned} \sum_{k=1}^n A_k \cos(2\pi f t + \phi_k) &= \sum_{k=1}^n \operatorname{Re}\{A_k e^{j2\pi f t + \phi_k}\} = \sum_{k=1}^n \operatorname{Re}\{A_k e^{j\phi_k} e^{j2\pi f t}\} \\ &= \left(\sum_{k=1}^n \operatorname{Re}\{A_k e^{j\phi_k}\}\right) e^{j2\pi f t} \end{aligned}$$

Ex. $3 \cos(2\pi 40t + \frac{\pi}{3}) - 1 \cos(2\pi 40t - \frac{\pi}{6}) + 2 \cos(2\pi 40t + \frac{\pi}{3})$

$$\begin{aligned} &\operatorname{Re}\left\{3e^{j\frac{\pi}{3}} e^{j2\pi 40t} - 1e^{-j\frac{\pi}{6}} e^{j2\pi 40t} + 2e^{j\frac{\pi}{3}} e^{j2\pi 40t}\right\} \\ &\operatorname{Re}\left\{\left(3e^{j\frac{\pi}{3}} - 1e^{-j\frac{\pi}{6}} + 2e^{j\frac{\pi}{3}}\right) e^{j2\pi 40t}\right\} \\ &\operatorname{Re}\{5.234 e^{j1.545} e^{j2\pi 40t}\} \\ &5.234 \cos(2\pi 40t + 1.545) \end{aligned}$$

multiply cosines of different frequency

$$\begin{aligned} &A_1 \cos(\omega_1 t) \cdot A_2 \cos(\omega_2 t + \phi) \\ &A_1 \left(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\right) A_2 \left(\frac{e^{j(\omega_2 t + \phi)} + e^{-j(\omega_2 t + \phi)}}{2}\right) \\ &\frac{A_1 A_2}{4} \left(e^{j\omega_1 t} e^{j(\omega_2 t + \phi)} + e^{j\omega_1 t} e^{-j(\omega_2 t + \phi)} + e^{-j\omega_1 t} e^{j(\omega_2 t + \phi)} + e^{-j\omega_1 t} e^{-j(\omega_2 t + \phi)}\right) \\ &\frac{A_1 A_2}{4} \left(e^{j(\omega_1 t + \omega_2 t + \phi)} + e^{-j(\omega_2 t - \omega_1 t + \phi)} + e^{j(\omega_2 t - \omega_1 t + \phi)} + e^{-j(\omega_1 t + \omega_2 t + \phi)}\right) \\ &\frac{A_1 A_2}{2} \left(\cos((\omega_1 + \omega_2)t + \phi) + \cos((\omega_2 - \omega_1)t + \phi)\right) \end{aligned}$$

Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

decompose a periodic signal x(t) into a sum of a series of sinusoids - the Fourier series.

Note: The sum of periodic functions is periodic.

ex.

$$X_k = \begin{cases} -8 & k \text{ odd} \\ \pi^2 k^2 & k \text{ even} \end{cases}$$

$$f_0 = 25 \text{ Hz}$$

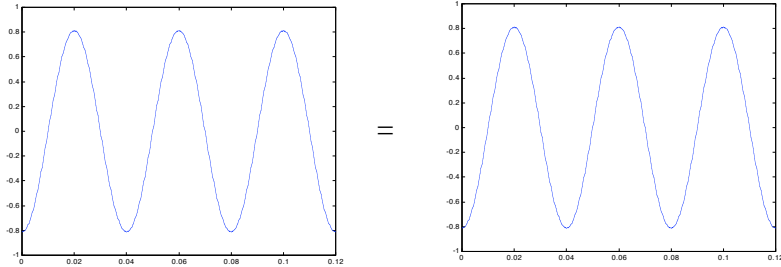
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_k = \begin{cases} \frac{-8}{\pi^2 k^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \quad X_k = \begin{cases} \frac{8}{\pi^2 k^2} e^{j\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

k=1

$$x(t) = 0.8105 \cos(2\pi 25t + \pi)$$



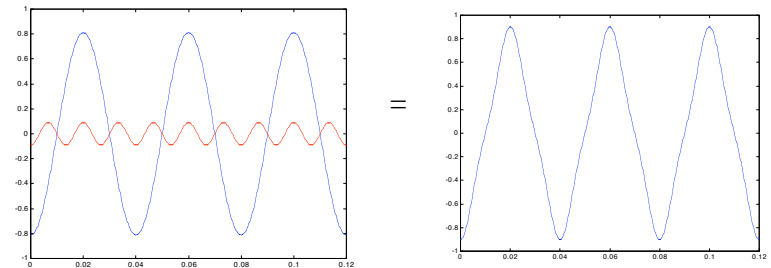
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_k = \begin{cases} \frac{8}{\pi^2 k^2} e^{j\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

k=3

$$x(t) = 0.8105 \cos(2\pi 25t + \pi) + 0.0901 \cos(2\pi 75t + \pi)$$



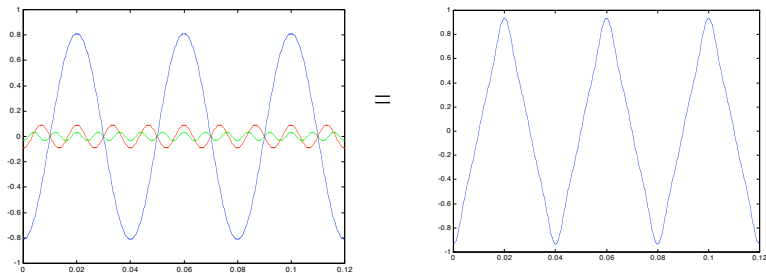
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

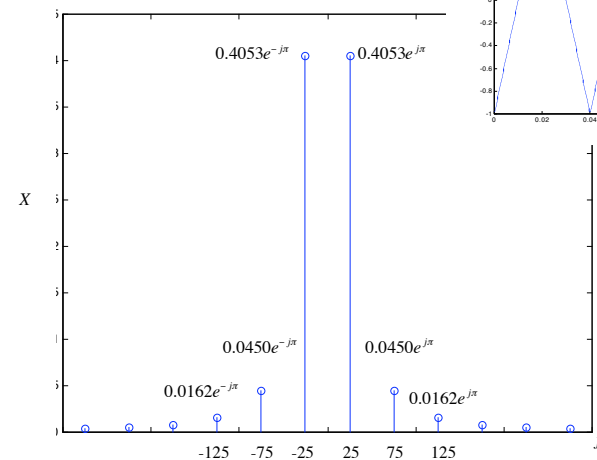
$$X_k = \begin{cases} \frac{8}{\pi^2 k^2} e^{j\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

k=5

$$x(t) = 0.8105 \cos(2\pi 25t + \pi) + 0.0901 \cos(2\pi 75t + \pi) + 0.0324 \cos(2\pi 125t + \pi)$$



spectrum



$$x(t) = 0.8105 \cos(2\pi 25t + \pi) + 0.0901 \cos(2\pi 75t + \pi) + 0.0324 \cos(2\pi 125t + \pi) + \dots$$

Fourier Series

For a given signal, how do we find $X_k = A_k e^{j\phi_k}$ for each k ?

Fourier Analysis

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

where

$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

f_0 :fundamental frequency
 $T_0 = 1/f_0$

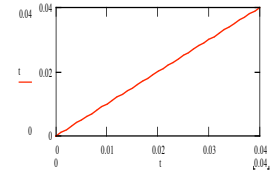
$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k t / T_0} dt$$

Fourier Series

$$x(t) = t \quad 0 \leq t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$\begin{aligned} X_0 &= \frac{1}{T_0} \int_0^{T_0} x(t) dt \\ &= \frac{1}{T_0} \int_0^{T_0} t dt = \frac{1}{T_0} \left. \frac{t^2}{2} \right|_0^{T_0} = \frac{1}{T_0} \frac{T_0^2}{2} = \frac{T_0}{2} \end{aligned}$$



Mathematica:

athena%add math

athena%math

In[1]:=1/T*Integrate[t,{t,0,T}]

Out[1]:=T/2

$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi k t / T_0} dt$$

Fourier Series

$$x(t) = t \quad 0 \leq t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{2}{T_0} \int_0^{T_0} t e^{-j2\pi k t / T_0} dt$$

$$X_k = \frac{T_0}{2} \frac{(j2\pi k + 1)}{\pi^2 k^2} e^{-2j\pi k} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = \frac{T_0}{2} \frac{(j2\pi k + 1)}{\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j \frac{T_0}{2} \frac{2k\pi}{\pi^2 k^2} + \frac{T_0}{2\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j \frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$\begin{aligned} e^{-2j\pi k} &= (e^{-j2\pi})^k \\ &= 1^k \\ &= 1 \end{aligned}$$

Fourier Series

Mathematica:

In[2]:= 2/T*Integrate[t*Exp[-I*2*Pi*k*t/T],{t,0,T}] $X_k = \frac{2}{T_0} \int_0^{T_0} t e^{-j2\pi k t / T_0} dt$

$$\text{Out[2]} = \frac{(((-1 + E^{(2 I) k \text{ Pi}} - (2 I) k \text{ Pi}) T))}{2 E^{(2 I) k \text{ Pi}} - 2 k \text{ Pi}}$$

In[3]:= Simplify[% ,Element[k,Integers]]

$$\begin{aligned} e^{-2j\pi k} &= 1 \\ e^{-j\pi k} &= -1^k \end{aligned}$$

$$\text{Out[3]} = \frac{I T}{k \text{ Pi}}$$

$$X_k = j \frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

Fourier Series

$$x(t) = t \quad 0 \leq t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

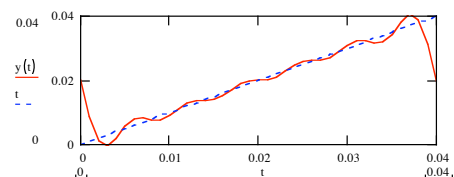
$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

f_0 : fundamental frequency
 $T_0 = 1/f_0$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos(2\pi k f_0 t + \frac{\pi}{2})$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos(2\pi f_0 t + \frac{\pi}{2}) + \frac{T_0}{2\pi} \cos(2\pi 2 f_0 t + \frac{\pi}{2}) + \dots$$



$f_0 = 25 \text{ Hz}$
 $T_0 = 1/f_0 = 0.04$
7 terms

Fourier Series

$$x(t) = t \quad 0 \leq t < T_0$$

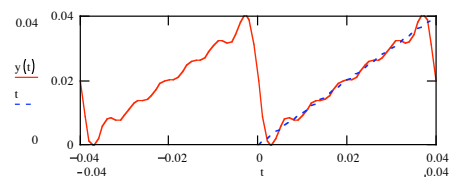
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \operatorname{Re} \left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos(2\pi k f_0 t + \frac{\pi}{2})$$

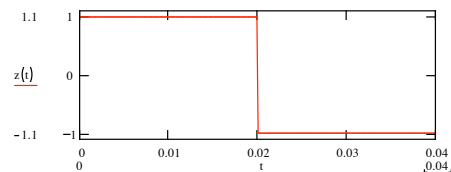
$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos(2\pi f_0 t + \frac{\pi}{2}) + \frac{T_0}{2\pi} \cos(2\pi 2 f_0 t + \frac{\pi}{2}) + \dots$$



Defined between $0 < t < 0.04$
Periodic with period 0.04

Fourier Series: Square Wave

$$x(t) = \begin{cases} 1 & 0 \leq t < T_0/2 \\ -1 & T_0/2 \leq t < T_0 \end{cases}$$



$$X_0 = \frac{1}{T_0} \int_0^{T_0/2} 1 dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} -1 dt$$

$$\text{In}[1] := 1/T_0 \int_0^{T_0/2} 1 dt + 1/T_0 \int_{T_0/2}^{T_0} -1 dt$$

$$\text{Out}[1] := 0$$

$$X_0 = 0$$

Fourier Series: Square Wave

$$X_k = \frac{2}{T_0} \int_0^{T_0/2} 1 e^{-j2\pi k t/T_0} dt + \frac{2}{T_0} \int_{T_0/2}^{T_0} -1 e^{-j2\pi k t/T_0} dt$$

$$\text{In}[2] := 2/T_0 \int_0^{T_0/2} \exp[-I*2*\text{Pi}*k*t/T_0], \{t, 0, T/2\} + 2/T_0 \int_{T_0/2}^{T_0} \exp[-I*2*\text{Pi}*k*t/T_0], \{t, T/2, T\}$$

$$\text{Out}[2] = \frac{-I k \text{Pi}}{k \text{Pi}} \frac{1 - E^{-I k \text{Pi}}}{E} + \frac{I k \text{Pi}}{(2 I) k \text{Pi}} \frac{1 - E^{-I k \text{Pi}}}{k \text{Pi}}$$

$$\text{In}[3] := \text{Simplify}[\%, \text{Element}[k, \text{Integers}]]$$

$$\text{Out}[7] = \frac{-I(-1 + (-1)^k)}{k \text{Pi}}$$

$$X_k = \begin{cases} -j \frac{4}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$X_k = \frac{-j(-1-1)^2}{k\pi} = -\frac{j(-2)^2}{k\pi} = -\frac{j4}{k\pi}$$

$$X_k = \frac{-j(-1+1)^2}{k\pi}$$

$$x(t)=\begin{cases} 1 & 0\leq t<T_0/2 \\ -1 & T_0/2\leq t<T_0 \end{cases}$$

$$X_0=0$$

$$X_k=\begin{cases} -j\frac{4}{k\pi} & k\text{ odd} \\ 0 & k\text{ even} \end{cases} \longrightarrow X_k=\begin{cases} \frac{4}{k\pi}e^{-j\frac{\pi}{2}} & k\text{ odd} \\ 0 & k\text{ even} \end{cases}$$

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi kf_0t+\phi_k)=X_0+\text{Re}\bigg\{\sum_{k=1}^{\infty}X_ke^{j2\pi kf_0t}\bigg\}$$

$$x(t)=\frac{4}{\pi}\cos\big(2\pi f_0t-\frac{\pi}{2}\big)+\frac{4}{3\pi}\cos\big(2\pi 3f_0t-\frac{\pi}{2}\big)+\dots$$

