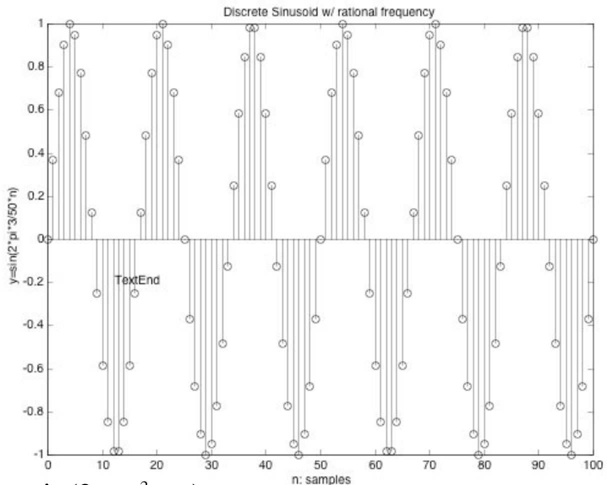
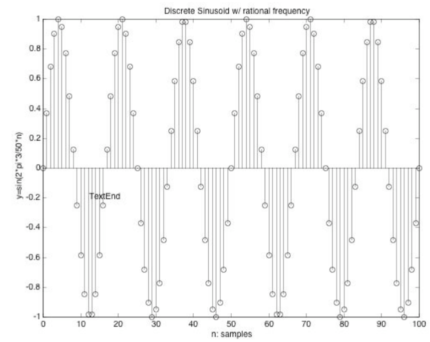


$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$

$y[n] = y[n+T]$ T=?? samples [integer] 50/3 ≠ integer



$y[n] = \sin(2\pi \cdot \frac{3}{50} \cdot n)$

$y[n] = y[n+T]$ T=?? samples

$\sin(0) = \sin(2\pi k)$

k=1,2…

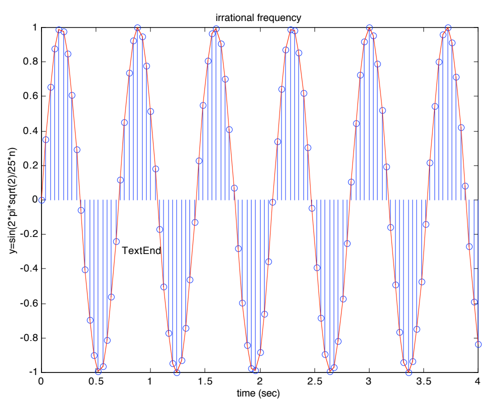
$2\pi \frac{3}{50} \cdot n = 2\pi k$

$\frac{n}{k} = \frac{50}{3} \frac{samples}{cycle}$ Ratio of integers

rational number

periodic

T=n=50 samples, k=3 cycles



$y(t) = \sin(2\pi \cdot \sqrt{2} \cdot t)$

$T = \frac{1}{\sqrt{2}}$ sec

continuous function
periodic

Ts=1/25 sec

$y[n] = \sin(2\pi \cdot \frac{\sqrt{2}}{25} \cdot n)$

$y[n] = y[n+T]$ T=?? samples

$\sin(0) = \sin(2\pi k)$ k=1,2…

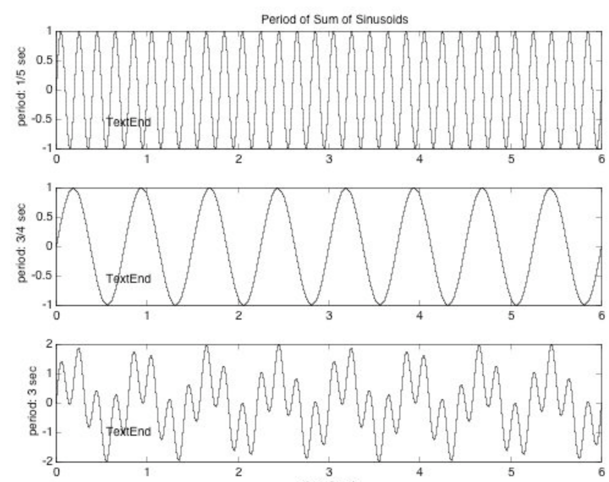
$2\pi \frac{\sqrt{2}}{25} \cdot n = 2\pi k$

$\frac{n}{k} = \frac{25\sqrt{2}}{2}$

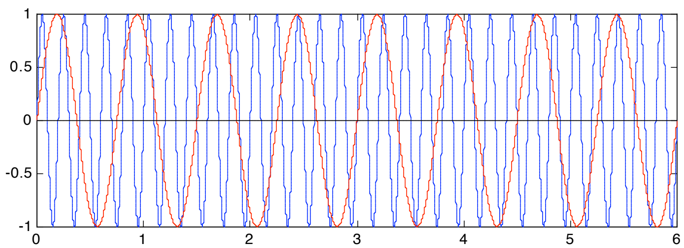
Equiv. discrete sinusoid not periodic

irrational number

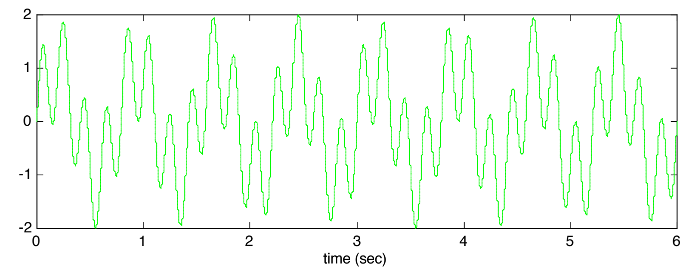
Period of Sum of Sinusoids



$y(t) = y(t+T)$



T1=0.2 seconds, T2=.75 seconds



Tsum=3 seconds

Least common multiple

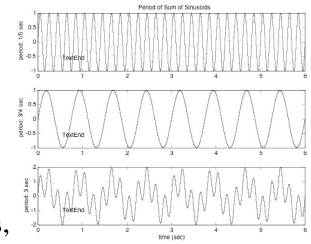
seconds to complete cycles

T1=1/5 seconds

1/5s, 2/5s, 3/5s …

4/20s, 8/20s, 12/20s, 16/20s, 20/20s, 24/20s, 28/20s, 32/20s, 36/20s, 40/20s, 44/20s, 48/20s, 52/20s, 56/20s, 60/20s

15 cycles



seconds to complete cycles

T2=3/4 seconds

3/4s, 6/4s, …

15/20s. 30/20s, 45/20s, 60/20s

4 cycles

1/5*k=3/4*l
k/l=15/4

rational number

Tsum=15*T1=15/5=3 seconds

Tsum=4*T2=3/4*4=3 seconds

Tsum=3 seconds

Complex Conversions

cartesian ⟶ polar

s=a+jb $\quad s=\sqrt{a^2+b^2}\,e^{j\cdot a\tan(b/a)}$

polar ⟶ cartesian

s=re$^{j\theta}$ $\quad s=r\cos\theta+jr\sin\theta$

Complex Arithmetic		
Addition	cartesian	$(a_1+jb_1)+(a_2+jb_2)=(a_1+a_2)+j(b_1+b_2)$
Subtraction	cartesian	$(a_1+jb_1)-(a_2+jb_2)=(a_1-a_2)+j(b_1-b_2)$
Multiplication	polar	$r_1e^{j\theta_1}\cdot r_2e^{j\theta_2}=r_1r_2e^{j(\theta_1+\theta_2)}$
Division	polar	$\dfrac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}}=\dfrac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$
Powers	polar	$\left(re^{j\theta}\right)^n=r^ne^{jn\theta}$
Roots	polar	$z^n=s=re^{j\theta}$ $z=s^{1/n}=r^{1/n}e^{j(\theta/n+2\pi k/n)}$ $\quad k=0,1,2\ldots n-1$

Complex Conversions

cartesian ⟶ polar

$3+j4=\sqrt{3^2+4^2}\,e^{j\cdot a\tan(4/3)}=5e^{j\cdot 0.927}$

polar ⟶ cartesian

$2e^{j\cdot\frac{\pi}{3}}=2\cos\frac{\pi}{3}+j2\sin\frac{\pi}{3}=1+j\sqrt{3}$

Complex Arithmetic		
Addition	cartesian	$(1+j2)+(3+j4)=(4+j6)$
Subtraction	cartesian	$(1+j2)-(3+j4)=(-2-j2)$
Multiplication	polar	$5e^{j\cdot\frac{\pi}{3}}\cdot 6e^{j\cdot\frac{\pi}{4}}=5\cdot 6e^{j\cdot\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}=30e^{j\cdot\frac{7\pi}{12}}$
Division	polar	$10e^{j\cdot\frac{\pi}{2}}\div 5e^{j\cdot\frac{\pi}{4}}=\left(\frac{10}{5}\right)e^{j\cdot\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}=2e^{j\cdot\frac{\pi}{4}}$
Powers	polar	$\left(3e^{j\cdot\frac{\pi}{4}}\right)^3=3^3\cdot e^{j\cdot\left(\frac{3\pi}{4}\right)}=27e^{j\cdot\left(\frac{3\pi}{4}\right)}$
Roots	polar	$z^3=64=64e^{j0}$ $z=64^{1/3}e^{j(0/3+2\pi k/3)}=4e^{j(2\pi k/3)}$ $\quad 4,\ 4e^{j(2\pi/3)},\ 4e^{j(4\pi/3)}$

Roots polar $z^n = s = re^{j\theta}$

$z = s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)}$ $k = 0,1,2\ldots n-1$

$\sqrt{4}$

2 $\quad -2$

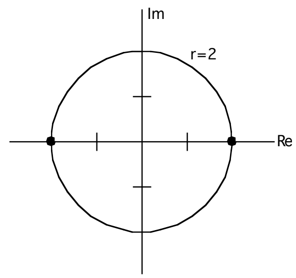
$2 \cdot 2 = 4$ $\quad -2 \cdot -2 = 4$

$\sqrt{4} = 4^{1/2}$

$\left(4e^{j0}\right)^{1/2} = 4^{1/2} e^{j(0/2 + 2\pi k/2)}$

$k=0$	$k=1$
$= 2e^{j(2\pi 0/2)}$	$= 2e^{j(2\pi/2)}$
$= 2e^{j0}$	$= 2e^{j\pi}$
$= 2 \cdot 1$	$= 2 \cdot -1$
$= 2$	$= -2$

$\sqrt{4} = \pm 2$ we expect n=2 solutions



Roots polar $z^n = s = re^{j\theta}$

$z = s^{1/n} = r^{1/n} e^{j(\theta/n + 2\pi k/n)}$ $k = 0,1,2\ldots n-1$

$\sqrt[4]{16} = 16^{1/4}$

2 $\quad -2$ $\quad ?$ $\quad ?$

$2 \cdot 2 \cdot 2 \cdot 2 = 16$ $\quad -2 \cdot -2 \cdot -2 \cdot -2 = 16$

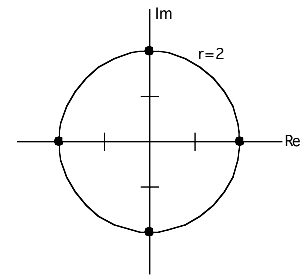
$\sqrt[4]{4} = 4^{1/4}$ we expect n=4 solutions

$\left(16e^{j0}\right)^{1/4} = 16^{1/4} e^{j(0/4 + 2\pi k/4)}$

$k=0$	$k=1$	$k=2$	$k=3$
$= 2e^{j(2\pi 0/4)}$	$= 2e^{j(2\pi 1/4)}$	$= 2e^{j(2\pi 2/4)}$	$= 2e^{j(2\pi 3/4)}$
$= 2e^{j0}$	$= 2e^{j\pi/2}$	$= 2e^{j\pi}$	$= 2e^{j3\pi/2}$
$= 2 \cdot 1$	$= 2 \cdot 1j$	$= 2 \cdot -1$	$= 2 \cdot -1j$
$= 2$	$= j2$	$= 2$	$= -j2$

$\sqrt{16} = \pm 2, \pm j2$

$-j2 \cdot -j2 \cdot -j2 \cdot -j2 = -2 \cdot -2 = 4$



Representations of Sinusoids

$$A\cos(2\pi k f_0 t + \phi_k)$$

$$\begin{aligned} &\mathrm{Re}\left\{Ae^{j2\pi f t + \phi}\right\} \\ &= \mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi f t}\right\} \\ &= \mathrm{Re}\left\{Xe^{j2\pi f t}\right\} \end{aligned}$$

$$\begin{aligned} &Ae^{j\phi} \cdot \left(\frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2}\right) \\ &= X \cdot \left(\frac{e^{j2\pi f t} + e^{-j2\pi f t}}{2}\right) \end{aligned}$$

Continuous Sinusoid

$y(t) = \cos(2\pi t)$

$\theta = 2\pi t$

$y(t) = \cos(\theta)$

Sample rate: $T_s = \frac{1}{8}\sec$

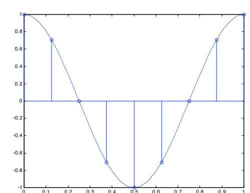
$t = nT_s$

Discrete Sinusoid

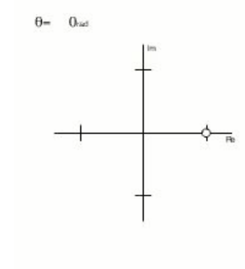
$y[n] = \cos(2\pi \cdot \frac{1}{8} \cdot n)$

$\theta = \frac{\pi}{4} n$

$y[n] = \cos(\theta)$



t(sec)	n	θ(rad)	cos(θ)
0	0	0	1
1/8	1	π/4	0.707
1/4	2	π /2	0
3/8	3	3 π /4	-0.707
1/2	4	π	-1
5/8	5	5 π /4	-0.707
3/4	6	3 π /2	0
7/8	7	7 π /4	0.707
1	8	2 π	1



$A\cos(2\pi f t + \phi)$ $\quad \cos(2\pi t)$

Continuous Sinusoid

$y(t)=\cos(2\pi t)$

$\theta=2\pi t$

$y(t)=\cos(\theta)$

Sample rate: $T_s=\frac{1}{8}$sec

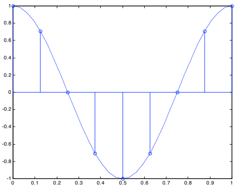
$t=nT_s$

Discrete Sinusoid

$y[n]=\cos(2\pi\cdot\frac{1}{8}\cdot n)$

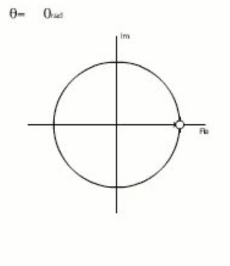
$\theta=\frac{\pi}{4}n$

$y[n]=\cos(\theta)$



t(sec)	n	θ(rad)	cos(θ)	exp(jθ)
0	0	0	1	1+0j
1/8	1	π/4	0.707	0.707+j0.707
1/4	2	π /2	0	0 + 1j
3/8	3	3 π /4	-0.707	-0.707+j0.707
1/2	4	π	-1	-1+0j
5/8	5	5 π /4	-0.707	-0.707-j0.707
3/4	6	3 π /2	0	0+j1
7/8	7	7 π /4	0.707	0.707-j0.707
1	8	2 π	1	1+j1

Rotating Phasor



$$A\cos(2\pi ft+\phi)$$
$$=\mathrm{Re}\left\{Ae^{j2\pi ft+\phi}\right\}$$
$$=\mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\}$$
$$=\mathrm{Re}\left\{Xe^{j2\pi ft}\right\}$$
$$X=Ae^{j\phi}$$

complex phasor

amplitude

$$\cos(2\pi t)$$
$$=\mathrm{Re}\left\{e^{j2\pi t}\right\}$$
$$X=1e^{j0}=1$$

Continuous Sinusoid

$y(t)=\cos(2\pi t+\frac{15\pi}{180})$

$\theta=2\pi t+\frac{15\pi}{180}$

$y(t)=\cos(\theta)$

Sample rate: $T_s=\frac{1}{8}$sec

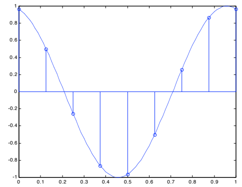
$t=nT_s$

Discrete Sinusoid

$y[n]=\cos(2\pi\cdot\frac{1}{8}\cdot n+\frac{15\pi}{180})$

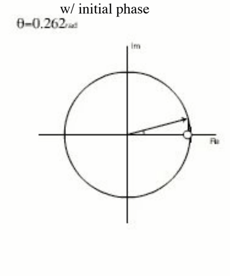
$\theta=\frac{\pi}{4}n+\frac{15\pi}{180}$

$y[n]=\cos(\theta)$



t(sec)	n	θ(rad)	cos(θ)	exp(jθ)
0	0	0.262	0.966	0.966-0.259j
1/8	1	1.047	0.500	0.500-j0.866
1/4	2	1.833	-0.259	-0.259+j0.966
3/8	3	2.618	-0.866	-0.866-j0.500
1/2	4	3.403	-0.96	-0.966+j0.259
5/8	5	4.189	-0.500	-0.500+j0.866
3/4	6	4.974	0.259	0.259+j0.966
7/8	7	5.760	0.866	0.866+j0.500
1	8	6.545	0.966	0.966-j0.259

Rotating Phasor
w/ initial phase



$$A\cos(2\pi ft+\phi)$$
$$=\mathrm{Re}\left\{Ae^{j2\pi ft+\phi}\right\}$$
$$=\mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\}$$
$$=\mathrm{Re}\left\{Xe^{j2\pi ft}\right\}$$
$$X=Ae^{j\phi}$$

complex phasor

amplitude

$$\cos(2\pi t+\tfrac{15\pi}{180})$$
$$=\mathrm{Re}\left\{e^{j\left(2\pi t+\frac{15\pi}{180}\right)}\right\}$$
$$=\mathrm{Re}\left\{e^{j\frac{15\pi}{180}}e^{j2\pi t}\right\}$$
$$=\mathrm{Re}\left\{Xe^{j2\pi t}\right\}$$
$$X=1e^{j\frac{15\pi}{180}}=e^{j\frac{15\pi}{180}}$$

Continuous Sinusoid

$y(t)=\cos(2\pi t)$

$\theta=2\pi t$

$y(t)=\cos(\theta)$

Sample rate: $T_s=\frac{1}{8}$sec

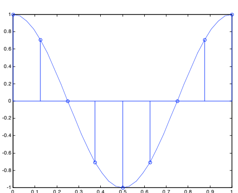
$t=nT_s$

Discrete Sinusoid

$y[n]=\cos(2\pi\cdot\frac{1}{8}\cdot n)$

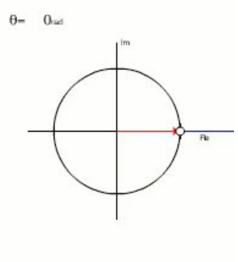
$\theta=\frac{\pi}{4}n$

$y[n]=\cos(\theta)$



t(sec)	n	θ(rad)	cos(θ)	exp(jθ)	exp(-jθ)
0	0	0	1	1+0j	1-0j
1/8	1	π/4	0.707	0.707+j0.707	0.707-j0.707
1/4	2	π /2	0	0 + 1j	0 - 1j
3/8	3	3 π /4	-0.707	-0.707+j0.707	-0.707-j0.707
1/2	4	π	-1	-1+0j	-1-0j
5/8	5	5 π /4	-0.707	-0.707-j0.707	-0.707+j0.707
3/4	6	3 π /2	0	0+j1	0-j1
7/8	7	7 π /4	0.707	0.707-j0.707	0.707+j0.707
1	8	2 π	1	1+j1	1-j1

complex conjugate pairs



$$A\cos(2\pi ft+\phi)$$
$$=A\left(\frac{e^{j(2\pi ft+\phi)}+e^{-j(2\pi ft+\phi)}}{2}\right)$$
$$=X\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$
$$X=Ae^{j\phi}$$

complex

amplitude

$$\cos(2\pi t)$$
$$=1\cdot\left(\frac{e^{j(2\pi t+0)}+e^{-j(2\pi t+0)}}{2}\right)$$
$$=X\cdot\left(\frac{e^{j2\pi t}+e^{-j2\pi t}}{2}\right)$$
$$X=1e^{j0}=1$$

Representations of Sinusoids

$$A\cos(2\pi kf_0t+\phi_k)$$

$$\mathrm{Re}\left\{Ae^{j2\pi ft+\phi}\right\}$$
$$=\mathrm{Re}\left\{Ae^{j\phi}e^{j2\pi ft}\right\}$$
$$=\mathrm{Re}\left\{Xe^{j2\pi ft}\right\}$$

$$Ae^{j\phi}\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$
$$=X\cdot\left(\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\right)$$

Sum multiple cosines same frequency

$$\sum_{k=1}^{n}A_k\cos(2\pi ft+\phi_k)=\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{2\pi ft+\phi_k}\right\}=\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{\phi_k}e^{2\pi ft}\right\}$$
$$=\left(\sum_{k=1}^{n}\mathrm{Re}\left\{A_ke^{\phi_k}\right\}\right)e^{2\pi ft}$$

Ex. $$3\cos(2\pi 40t+\tfrac{\pi}{2})-1\cos(2\pi 40t-\tfrac{\pi}{6})+2\cos(2\pi 40t+\tfrac{\pi}{3})$$
$$3e^{j\frac{\pi}{2}}e^{j2\pi 40t}-1e^{-j\frac{\pi}{6}}e^{2\pi 40t}+2e^{\frac{\pi}{3}}e^{2\pi 40t}$$
$$\left(3e^{j\frac{\pi}{2}}-1e^{-j\frac{\pi}{6}}+2e^{\frac{\pi}{3}}\right)e^{2\pi 40t}$$
$$5.234e^{j1.545}e^{2\pi 40t}$$
$$5.234\cos(2\pi 40t+1.545)$$

multiply cosines of different frequency

$$A_1\cos(\omega_1 t)\cdot A_2\cos(\omega_2 t+\phi)$$

$$A_1\left(\frac{e^{j\omega_1 t}+e^{-j\omega_1 t}}{2}\right)A_2\left(\frac{e^{j(\omega_2 t+\phi)}+e^{-j(\omega_2 t+\phi)}}{2}\right)$$

$$\frac{A_1A_2}{4}\left(e^{j\omega_1 t}e^{j(\omega_2 t+\phi)}+e^{j\omega_1 t}e^{-j(\omega_2 t+\phi)}+e^{-j\omega_1 t}e^{j(\omega_2 t+\phi)}+e^{-j\omega_1 t}e^{-j(\omega_2 t+\phi)}\right)$$

$$\frac{A_1A_2}{4}\left(e^{j(\omega_1 t+\omega_2 t+\phi)}+e^{-j(\omega_2 t-\omega_1 t+\phi)}+e^{j(\omega_2 t-\omega_1 t+\phi)}+e^{-j(\omega_1 t+\omega_2 t+\phi)}\right)$$

$$\frac{A_1A_2}{2}\left(\cos\left((\omega_1+\omega_2)t+\phi\right)+\cos\left((\omega_2-\omega_1)t+\phi\right)\right)$$

Composite signals (waveform synthesis)

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi f_k t+\phi_k)=X_0+\mathrm{Re}\left\{\sum_{k=1}^{\infty}X_k e^{j2\pi f_k t}\right\}$$

Consider harmonic signals

$f_k = kf_0$ Harmonics

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi kf_0 t+\phi_k)=X_0+\mathrm{Re}\left\{\sum_{k=1}^{\infty}X_k e^{j2\pi kf_0 t}\right\}$$

Composite signals (waveform synthesis)

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi kf_0 t+\phi_k)=X_0+\mathrm{Re}\left\{\sum_{k=1}^{\infty}X_k e^{j2\pi kf_0 t}\right\}$$

decompose a periodic signal x(t) into a sum of a series of sinusoids - the Fourier series.

Note: The sum of periodic functions is periodic.

ex.

$$X_k=\begin{cases}\dfrac{-8}{\pi^2k^2} & k\ \ odd\\ 0 & k\ \ even\end{cases}$$

$$f_0 = 25Hz$$

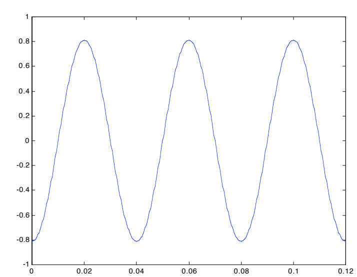
Composite signals (waveform synthesis)

$$x(t)=A_0+\sum_{k=1}^{\infty}A_k\cos(2\pi kf_0 t+\phi_k)=X_0+\mathrm{Re}\left\{\sum_{k=1}^{\infty}X_k e^{j2\pi kf_0 t}\right\}$$

$$X_k=\begin{cases}\dfrac{-8}{\pi^2k^2} & k\ \ odd\\ 0 & k\ \ even\end{cases}\qquad X_k=\begin{cases}\dfrac{8}{\pi^2k^2}e^{j\pi} & k\ \ odd\\ 0 & k\ \ even\end{cases}$$

k=1

$$x(t)=0.8105\cos(2\pi 25t+\pi)$$



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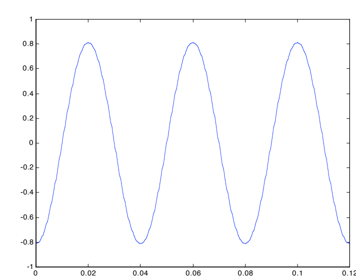


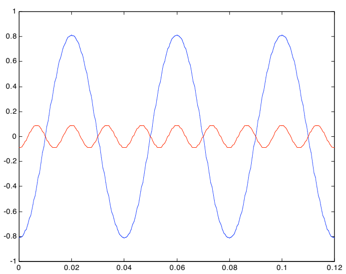
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=3

$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi)$$



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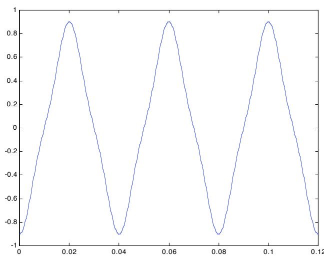


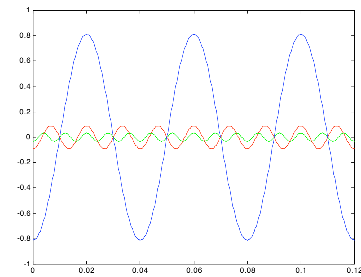
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=5

$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi)$$



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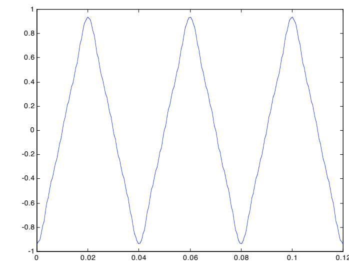
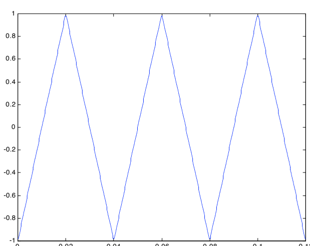
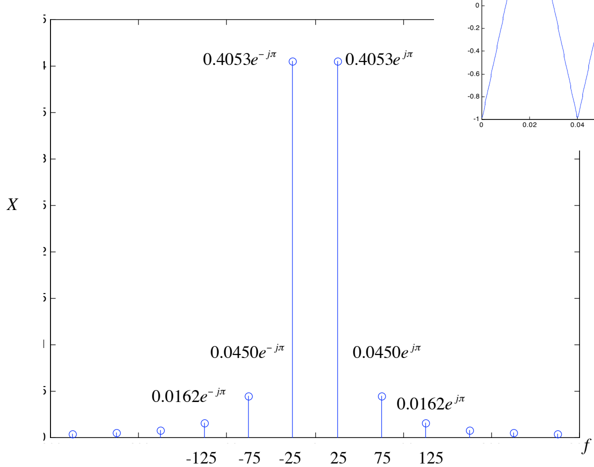


spectrum





$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi) + \ldots$$

Fourier Series

For a given signal, how do we find $X_k = A_k e^{j\phi_k}$ for each k ?

Fourier Analysis

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

where

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t) e^{-j2\pi kt/T_0}\,dt$$

f_0:fundamental frequency

$T_0 = 1/f_0$

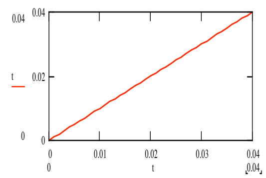
Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\,dt$$

$$= \frac{1}{T_0} \int_0^{T_0} t\,dt = \frac{1}{T_0} \left. \frac{t^2}{2} \right|_0^{T_0} = \frac{1}{T_0} \frac{T_0^{\,2}}{2} = \frac{T_0}{2}$$



Mathematica:
athena%add math
athena%math
In[1]:=1/T*Integrate[t,{t,0,T}]
Out[1]:=T/2

$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{2}{T_0} \int_0^{T_0} t e^{-j2\pi kt/T_0} dt$$

$$\int x e^{cx} dx = \frac{e^{cx}}{c^2}(cx - 1)$$

$$c = \frac{-j2\pi k}{T_0}$$

$$X_k = \frac{2}{T_0} \left(\frac{e^{-j2\pi k T_0/T_0}}{(-j2\pi k/T_0)^2} \right) (-j2\pi k T_0/T_0 - 1) - \frac{2}{T_0} \left(\frac{1}{(-j2\pi k/T_0)^2} \right) (-1)$$

$$X_k = \frac{T_0}{2} \frac{(j2\pi k + 1)}{\pi^2 k^2} e^{-2j\pi k} - \frac{T_0}{2\pi^2 k^2}$$

$$e^{-2j\pi k} = \left(e^{-j2\pi}\right)^k$$
$$= 1^k$$
$$= 1$$

$$X_k = \frac{T_0}{2} \frac{(j2\pi k + 1)}{\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j \frac{T_0}{2} \frac{2k\pi}{\pi^2 k^2} + \frac{T_0}{2\pi^2 k^2} - \frac{T_0}{2\pi^2 k^2}$$

$$X_k = j \frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

Fourier Series

Mathematica:
In[2]:= 2/T*Integrate[t*Exp[-I*2*Pi*k*t/T],{t,0,T}]

$$X_k = \frac{2}{T_0} \int_0^{T_0} t e^{-j2\pi kt/T_0} dt$$

```
                 (2 I) k Pi
        -((-1 + E           - (2 I) k Pi) T)
Out[2]= ------------------------------------
                (2 I) k Pi  2   2
             2 E            k  Pi
```

$$X_k = \frac{-\left(\left(-1 + e^{2jk\pi} - 2jk\pi\right)T\right)}{2e^{2jk\pi} k^2 \pi^2}$$

In[3]:= Simplify[%,Element[k,Integers]]

$$e^{-2j\pi k} = 1$$
$$e^{-j\pi k} = -1^k$$

```
         I T
Out[3]= ----
         k Pi
```

$$X_k = j \frac{T_0}{\pi k} = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

Fourier Series

$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t} \right\}$$

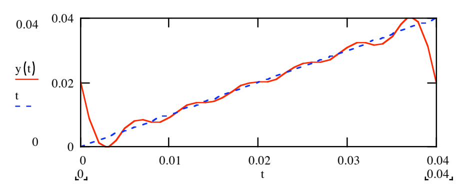
$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

f_0:fundamental frequency
$T_0 = 1/f_0$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos(2\pi k f_0 t + \tfrac{\pi}{2})$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos(2\pi f_0 t + \tfrac{\pi}{2}) + \frac{T_0}{2\pi} \cos(2\pi 2 f_0 t + \tfrac{\pi}{2}) + \ldots$$



$f_0 = 25Hz$
$T_0 = 1/f_0 = 0.04$
7 terms

Fourier Series

$x(t) = t \quad 0 \le t < T_0$

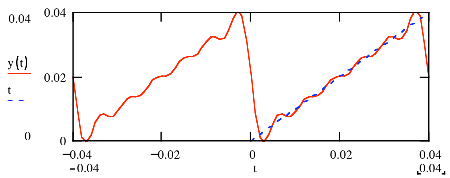
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2}$$

$$X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos\left(2\pi k f_0 t + \tfrac{\pi}{2}\right)$$

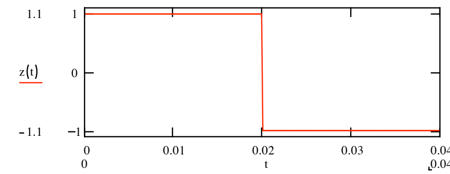
$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos\left(2\pi f_0 t + \tfrac{\pi}{2}\right) + \frac{T_0}{2\pi} \cos\left(2\pi 2 f_0 t + \tfrac{\pi}{2}\right) + \dots$$



Defined between 0<t<0.04
Periodic with period 0.04

Fourier Series:Square Wave

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$



$$X_0 = \frac{1}{T_0}\int_0^{T_0/2} 1\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -1\,dt$$

In[1]:=1/T*Integrate[1,{t,0,T/2}]+ 1/T*Integrate[-1,{t,T/2,T}]
Out[1]:=0

$$X_0 = 0$$

Fourier Series:Square Wave

$$X_k = \frac{2}{T_0}\int_0^{T_0/2} 1 e^{-j2\pi k t/T_0} dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} -1 e^{-j2\pi k t/T_0} dt$$

```
In[2]:=2/T*Integrate[Exp[-I*2*Pi*k*t/T],{t,0,T/2}]+
        2/T*Integrate[- Exp[-I*2*Pi*k*t/T],{t,T/2,T}]
                 -I k Pi              I k Pi
        -I (1 - E        )   I (-1 + E        )
Out[2]= ----------------- + ----------------
        k Pi                   (2 I) k Pi
                              E           k Pi
```

$$X_k = \frac{-j\left(1-e^{-jk\pi}\right)}{k\pi} + \frac{j\left(-1+e^{jk\pi}\right)}{e^{2jk\pi} k\pi}$$

In[3]:= Simplify[%,Element[k,Integers]]

```
                  k 2
        -I (-1 + (-1) )
Out[7]= ---------------
             k Pi
```

$$X_k = \frac{-j\left(-1+(-1)^k\right)^2}{k\pi}$$

$$X_k = \begin{cases} -j\frac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \frac{-j(-1-1)^2}{k\pi} = -\frac{j(-2)^2}{k\pi} = -\frac{j4}{k\pi}$$

$$X_k = \frac{-j(-1+1)^2}{k\pi}$$

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

$$X_0 = 0$$

$$X_k = \begin{cases} -j\frac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases} \longrightarrow X_k = \begin{cases} \frac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \dots$$