

What do you do with negative amplitudes?

Amplitudes always positive

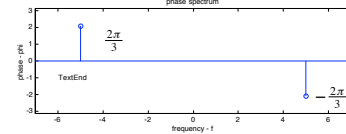
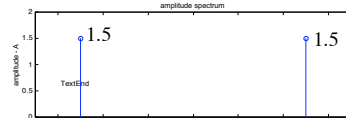
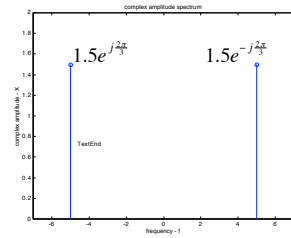
$$-C \cos(2\pi ft) = C \cos(2\pi ft \pm \pi)$$

Absorb the minus sign into the angle

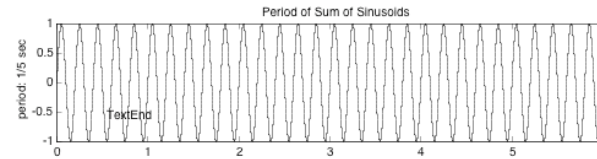
$$\begin{aligned} -3 \cos(2\pi \cdot 5t + \frac{\pi}{3}) &= 3 \cos(2\pi \cdot 5t + \frac{\pi}{3} - \pi) \\ &= 3 \cos(2\pi \cdot 5t - \frac{2\pi}{3}) \end{aligned}$$

$$\begin{aligned} \cos(2\pi ft + \phi) &= \frac{e^{j2\pi ft + \phi} + e^{-j2\pi ft + \phi}}{2} \\ &= \frac{e^{j\phi} e^{j2\pi ft} + e^{-j\phi} e^{-j2\pi ft}}{2} \\ &= X_{km} e^{j2\pi ft} + X_{kp} e^{-j2\pi ft} \quad X_{km} = \frac{1}{2} e^{j\phi} \\ &= \sum X_k e^{j2\pi ft} \\ X &= \begin{cases} \frac{1}{2} e^{j\phi} & k = f \\ \frac{1}{2} e^{-j\phi} & k = -f \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

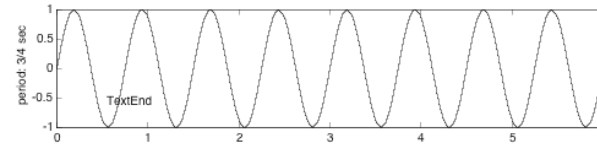
ϕ always a odd pair



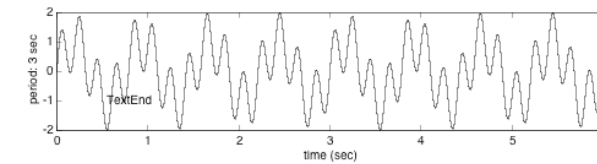
Period of Sum of Sinusoids



$$x(t) = \cos(2\pi 5t)$$



$$y(t) = \cos(2\pi (\frac{1}{3})t)$$



$$z(t) = x(t) + y(t)$$

$$z(t) = z(t + T), T = ?$$

$$x(t) = \cos(2\pi 5t)$$

Least common multiple

$$y(t) = \cos(2\pi (\frac{1}{3})t)$$

seconds to complete cycle

$$T_x = 1/5 \text{ seconds}$$

1/5s, 2/5s, 3/5s ...

4/20s, 8/20s, 12/20s,
16/20s, 20/20s, 24/20s,
28/20s, 32/20s, 36/20s,
40/20s, 44/20s, 48/20s,
52/20s, 56/20s, 60/20s

15 cycles

$$1/5 * k = 3/4 * l$$

$$k/l = 15/4$$

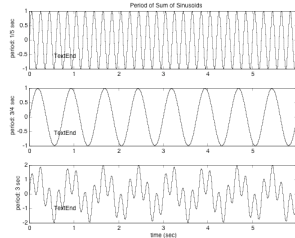
rational number

$$z(t) = x(t) + y(t)$$

$$T_z = 15 * T_x = 15/5 = 3 \text{ seconds} \quad T_z = 4 * T_y = 3/4 * 4 = 3 \text{ seconds}$$

$$z(t) = z(t + T_z)$$

$$T_z = 3 \text{ seconds}$$



seconds to complete cycle

$$T_y = 3/4 \text{ seconds}$$

3/4s, 6/4s, ...

15/20s, 30/20s,
45/20s, 60/20s

4 cycles

Fourier Series

$$x(t) = \sin(2\pi t) \quad 0 \leq t < 1$$

$$= 0 + 1 \cdot \cos\left(2\pi t - \frac{\pi}{2}\right) + 0 \cdot \cos\left(2\pi 2t - \frac{\pi}{2}\right) + 0 \cdot \cos\left(2\pi 3t - \frac{\pi}{2}\right) + \dots$$

$$X_0 = 0$$

$$X_k = \begin{cases} 1e^{-j\pi/2} & k = 1 \\ 0 & k \neq 1 \end{cases}$$

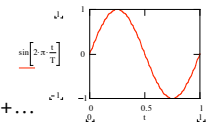
$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

$$= \int_0^1 \sin(2\pi t) dt = 0$$

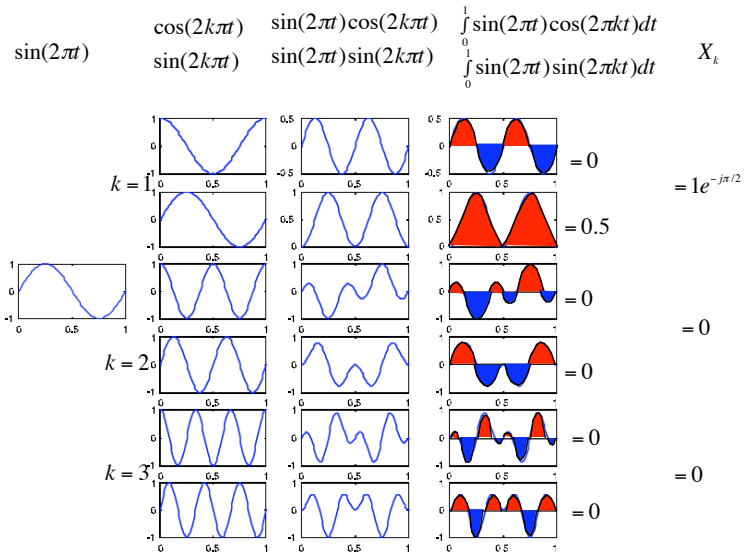
$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi \frac{k}{T_0} t} dt$$

$$= 2 \int_0^1 \sin(2\pi t) e^{-j2\pi kt} dt$$

$$= 2 \int_0^1 \sin(2\pi t) [\cos(-2\pi kt) + j \sin(-2\pi kt)] dt$$



Fourier Series



Fourier Series (frequency space)

$X_0 = \frac{1}{T} \int_0^T z(t) dt$
 $X_k = \frac{2}{T} \int_0^T z(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{t}{T}} dt$

$X_0 = \frac{1}{3} \int_0^3 \cos(2 \cdot \pi \cdot 5 \cdot t) + \cos\left[2 \cdot \pi \cdot \left(\frac{4}{3}\right) \cdot t\right] dt$
 $X_k = \frac{2}{3} \int_0^3 \left(\cos(2 \cdot \pi \cdot 5 \cdot t) + \cos\left(2 \cdot \pi \cdot \frac{4}{3} \cdot t\right) \right) \cdot e^{-j \cdot 2 \cdot \pi \cdot k \cdot \frac{t}{3}} dt$

$X_0 = 0$
 $X_k = i \cdot \frac{(2 \cdot \exp(-2 \cdot i \cdot \pi \cdot k) \cdot k^3 - 2 \cdot k^3 - 241 \cdot \exp(-2 \cdot i \cdot \pi \cdot k) \cdot k + 241 \cdot k)}{(\pi \cdot k^4 - 241 \cdot \pi \cdot k^2 + 3600 \cdot \pi)}$

$k=4, k=15$
 $k \neq 4, 15$

$X_{4,15} = \frac{0}{0}$
 $X_k = i \cdot \frac{2 \cdot 1 \cdot k^3 - 2 \cdot k^3 - 241 \cdot 1 \cdot k + 241 \cdot k}{[\pi \cdot (k+15) \cdot (k-4) \cdot (k+4) \cdot (k-15)]}$

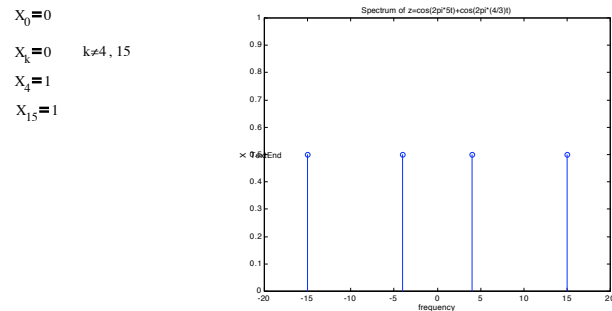
$X_k = 0$

So, use L'Hopitals Rule

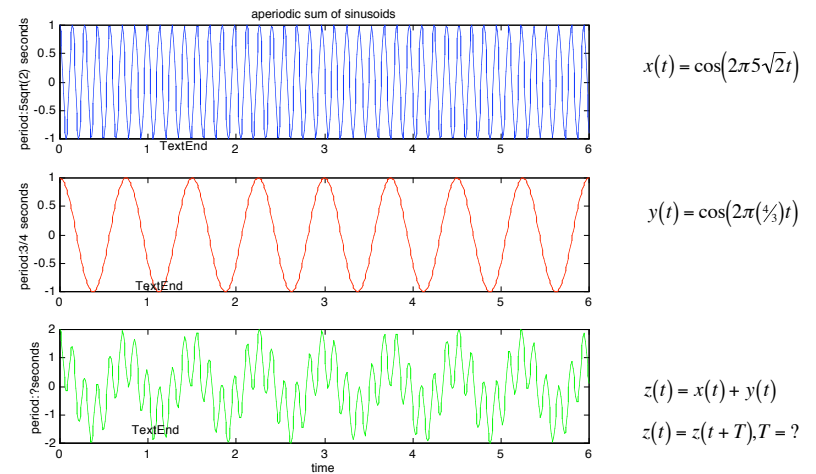
$X_k = \frac{1}{4} \cdot i \cdot \frac{-4 \cdot 1 \cdot \pi \cdot \exp(-2 \cdot 1 \cdot \pi \cdot k) \cdot k^3 + 6 \cdot \exp(-2 \cdot 1 \cdot \pi \cdot k) \cdot k^2 - 6 \cdot k^2 + 482 \cdot \pi \cdot \exp(-2 \cdot 1 \cdot \pi \cdot k) \cdot k - 241 \cdot \exp(-2 \cdot 1 \cdot \pi \cdot k) + 241}{\pi \cdot (k-4) \cdot (k+4) \cdot (k-15) + \pi \cdot (k+15) \cdot (k+4) \cdot (k-15) + \pi \cdot (k+15) \cdot (k-4) \cdot (k-15) + \pi \cdot (k+15) \cdot (k-4) \cdot (k+4)}$

$X_4 = 1$
 $X_{15} = 1$

Fourier Series (frequency space)



Aperiodic Sum of Sinusoids w/ an Irrational Frequency



$$x(t) = \cos(2\pi 5\sqrt{2}t)$$

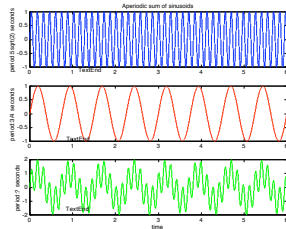
Least common multiple

$$y(t) = \cos(2\pi(\frac{1}{5})t)$$

seconds to complete cycle

$$T_x = \frac{1}{5\sqrt{2}} \text{ seconds}$$

$$\frac{1}{5\sqrt{2}}s, \frac{2}{5\sqrt{2}}s, \frac{3}{5\sqrt{2}}s, \dots$$



seconds to complete cycle

$$T_y = \frac{3}{4} \text{ seconds}$$

$$\frac{3}{4}s, \frac{6}{4}s, \frac{9}{4}s, \dots$$

$$\frac{1}{5\sqrt{2}}k = \frac{3}{4}l$$

$$\frac{k}{l} = \frac{15\sqrt{2}}{4}$$

irrational number

$$z(t) = x(t) + y(t)$$

$$z(t) = z(t + T_z)$$

$$T_z = \infty \text{ seconds}$$

$z(t)$ aperiodic

QUAD8 Numerically evaluate integral, higher order method.

Q = QUAD8(F,A,B) approximates the integral of F(X) from A to B
F is a string containing the name of the function.

The function must return a vector of output values given a vector of input values.

Q = QUAD8(F,A,B,TOL,TRACE,P1,P2,...) allows coefficients P1, P2, ...
to be passed directly to function F: G = F(X,P1,P2,...).

To use default values for TOL or TRACE, you may pass in the empty
matrix ([]).

$$x(t) = \int_0^t \cos^2(2\pi ft) dt$$

function y = myintegrand(t,f)
y=cos(2*pi*f*t).^2;

save as myintegrand.m

```
>>f=1;
>>quad8('myintegrand',0,1,[],[],1)
ans =
    0.500000000000000
```

$$X_0 = \frac{1}{T} \int_0^T z(t) dt \quad X_k = \frac{2}{T} \int_0^T z(t) \cdot e^{-1j \cdot 2 \cdot \pi \cdot k \cdot \frac{t}{T}} dt$$

Pick a T, compute X0, loop over Xk's, plot the spectrum, then repeat with larger T's.

Fourier Series for Irrational Frequency

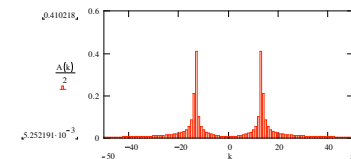
“What’s the frequency, Kenneth?”

In 1986, CBS Anchorman Rather was confronted about 11 p.m. while walking on Park Avenue, when he was punched from behind and knocked to the ground then chased into a building and kicked him several times in the back while the assailant demanded to know 'Kenneth, what is the frequency?' The assailant was convinced the media had him under surveillance and were beaming hostile messages into his head, and he demanded that Rather tell him the frequency being used.

In the Fourier Series for an aperiodic signal

“what’s the period, Quinn?”

$$X_0 = \frac{1}{T} \int_0^T z(t) dt \quad X_k = \frac{2}{T} \int_0^T z(t) \cdot e^{-1j \cdot 2 \cdot \pi \cdot k \cdot \frac{t}{T}} dt$$



PS3-5

Pick a T, plot the spectrum, then repeat with larger T's.

Bases are building blocks to form more complex things

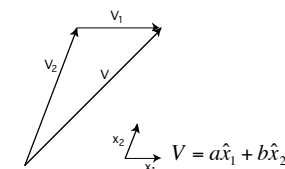
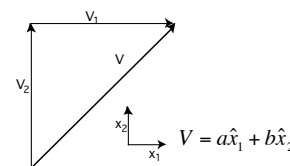
$$x = \sum_{k=1}^m A_k \phi_k$$

Synthesize x from a weighted sum of basis elements, ϕ

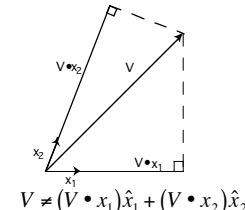
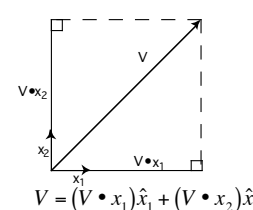
or

Decompose x into a weighted sum of basis elements, ϕ

Basis Vector



Prefer orthonormal basis vectors



projections

Basis Functions

$$x(t) = \sum_k A_k \phi_k(t) \quad x(t), \phi_k(t) \text{ are functions}$$

When are functions orthogonal to each other?

$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \begin{array}{l} \text{normalized} \\ \text{orthogonality condition} \end{array} \quad \begin{array}{l} \text{over an interval} \\ a \leq t \leq b \end{array}$$

How do you “project” functions onto each other?

$$A_k = \int_a^b x(t) \phi_k^* dt \quad \text{How much of } \phi_k(t) \text{ is in } x(t)?$$

If $\phi_k(t)$ are orthonormal:

$$x(t) = \sum_k A_k \phi_k(t) \quad \text{where} \quad A_k = \int_a^b x(t) \phi_k^* dt$$

Fourier Series

$$\phi_k = \frac{1}{\sqrt{T_0}} e^{j2\pi k t / T_0}$$

Complex exponentials

Basis Functions

$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \begin{array}{l} \text{normalized} \\ \text{orthogonality condition} \end{array}$$

over an interval
 $a \leq t \leq b$

Ex:

$$\phi_1 = \frac{\sqrt{6}}{3} \cos(2\pi t) \quad \int_a^b \phi_1 \phi_2 dt = \int_0^3 \frac{\sqrt{6}}{3} \cos(2\pi t) \frac{\sqrt{6}}{3} \cos(2\pi(\frac{1}{3})t) dt \quad \text{orthogonal}$$

$$\phi_2 = \frac{\sqrt{6}}{3} \cos(2\pi(\frac{1}{3})t)$$

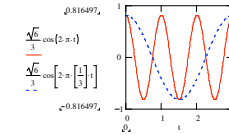
over an interval
 $0 \leq t \leq 3$

$$\int_a^b \phi_1 \phi_1 dt = \int_0^3 \frac{\sqrt{6}}{3} \cos(2\pi t) \frac{\sqrt{6}}{3} \cos(2\pi t) dt \quad \text{normalized}$$

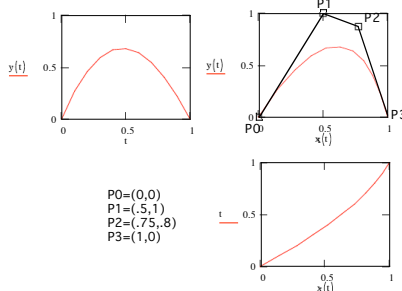
$$= 1$$

$$\int_a^b \phi_2 \phi_2 dt = \frac{1}{3} \int_0^3 \frac{\sqrt{6}}{3} \cos(2\pi(\frac{1}{3})t) \frac{\sqrt{6}}{3} \cos(2\pi(\frac{1}{3})t) dt \quad \text{normalized}$$

$$= 1$$



Bezier Curves



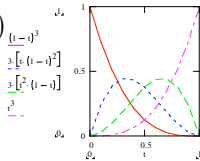
cubic spline basis function

$$\phi_1 = (t-1)^3 \quad 0 \leq t \leq 1$$

$$\phi_2 = 3t(t-1)^2$$

$$\phi_3 = 3t^2(t-1)$$

$$\phi_4 = t^3$$



$$x(t) = P0_x(1-t)^3 + 3P1_x(1-t)^2t + 3P2_x(1-t)t^2 + P3_x(1-t)^3$$

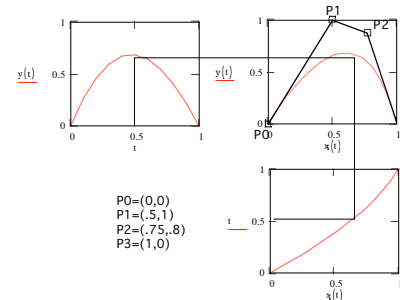
$$y(t) = P0_y(1-t)^3 + 3P1_y(1-t)^2t + 3P2_y(1-t)t^2 + P3_y(1-t)^3 \quad \text{parametric curve}$$

$$\phi_1 = (t-1)^3 \quad \int_a^b \phi_1 \phi_2 dt = \int_0^1 (t-1)^3 3t(t-1)^2 dt \quad \text{not orthogonal}$$

$$\phi_2 = 3t(t-1)^2$$

$$= -\frac{1}{14}$$

Bezier Curves



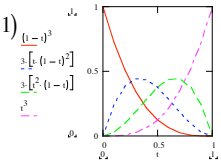
cubic spline basis function

$$\phi_1 = (t-1)^3 \quad 0 \leq t \leq 1$$

$$\phi_2 = 3t(t-1)^2$$

$$\phi_3 = 3t^2(t-1)$$

$$\phi_4 = t^3$$



$$x(t) = P0_x(1-t)^3 + 3P1_x(1-t)^2t + 3P2_x(1-t)t^2 + P3_x(1-t)^3$$

$$y(t) = P0_y(1-t)^3 + 3P1_y(1-t)^2t + 3P2_y(1-t)t^2 + P3_y(1-t)^3 \quad \text{parametric curve}$$

$$\phi_1 = (t-1)^3 \quad \int_a^b \phi_1 \phi_2 dt = \int_0^1 (t-1)^3 3t(t-1)^2 dt \quad \text{not orthogonal}$$

$$\phi_2 = 3t(t-1)^2$$

$$= -\frac{1}{14}$$

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 and an interval $a \leq t \leq b$
2. Normalize ϕ_1

$$\int_a^b \phi_1 \phi_1 dt = 1$$
 1 eqn
3. Pick a general formula for a second function ϕ_2
4. "Orthonormalize"
 a. make ϕ_2 orthogonal to ϕ_1

$$\int_a^b \phi_1 \phi_2 dt = 0$$

 b. normalize ϕ_2

$$\int_a^b \phi_2 \phi_2 dt = 1$$
 2 eqns
5. Pick a general formula for a second function ϕ_3
6. "Orthonormalize"
 a. make ϕ_3 orthogonal to ϕ_1 & ϕ_2

$$\int_a^b \phi_3 \phi_1 dt = 0$$

$$\int_a^b \phi_3 \phi_2 dt = 0$$

 b. normalize ϕ_3

$$\int_a^b \phi_3 \phi_3 dt = 1$$
 3 eqns
7. Continue for ϕ_k k eqns

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 with one parameter, & interval $a \leq t \leq b$
2. Normalize ϕ_1

$$\int_a^b \phi_1 \phi_1 dt = 1$$
 1 eqn
3. Pick a general formula for a second function ϕ_2 with 2 parameters & 1 zero-crossing
4. "Orthonormalize"
 a. make ϕ_2 orthogonal to ϕ_1

$$\int_a^b \phi_1 \phi_2 dt = 0$$

 b. normalize ϕ_2

$$\int_a^b \phi_2 \phi_2 dt = 1$$
 2 eqns
5. Pick a general formula for a second function ϕ_3 with 3 parameters & 2 zero-crossings
6. "Orthonormalize"
 a. make ϕ_3 orthogonal to ϕ_1 & ϕ_2

$$\int_a^b \phi_3 \phi_1 dt = 0$$

$$\int_a^b \phi_3 \phi_2 dt = 0$$

 b. normalize ϕ_3

$$\int_a^b \phi_3 \phi_3 dt = 1$$
 3 eqns
7. Continue for ϕ_k with k parameters k eqns

Make your own set of basis functions

1. Pick an initial basis function ϕ_1 with one parameter & an interval

$$\phi_1 = A \quad 0 \leq t \leq 1$$

2. Normalize $\int_0^1 \phi_1 \phi_1 dt = 1$

$$\int_0^1 A A dt = 1$$

$$A^2 t \Big|_0^1 = 1$$

$$A^2 = 1$$

$$A = 1$$

$$\phi_1 = 1$$

3. Pick an second basis function ϕ_2 with two parameters
 $\phi_2 = Bt + C$

4. Orthonormalize

$$\int_0^1 \phi_2 \phi_2 dt = 1$$

$$\int_0^1 (Bt + C)(Bt + C) dt = 1$$

$$B^2 t^3 / 3 + 2CBt^2 / 2 + C^2 t \Big|_0^1 = 1$$

$$\frac{B^2}{3} + BC + C^2 = 1$$

$$\int_0^1 \phi_2 \phi_1 dt = 0$$

$$\int_0^1 (Bt + C) 1 dt = 0$$

$$\frac{B}{2} + C = 0$$

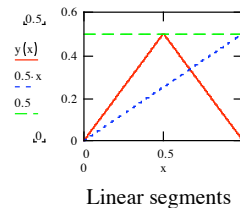
$$C = -\frac{B}{2}$$

$$\frac{B^2}{3} - B \frac{B^2}{2} + \left(-\frac{B}{2}\right)^2 = 1$$

$$B = \pm 2\sqrt{3}$$

$$C = \mp \sqrt{3}$$

$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$



$$\phi_1 = 1$$

$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$

5. Pick an third basis function ϕ_3 with three parameters

$$\phi_3 = \begin{cases} bt + a & t \leq c \\ (bc + a) - b(t - c) & t > c \end{cases}$$

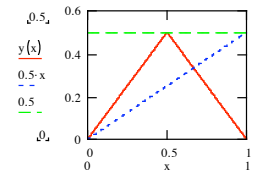
6. Orthonormalize

$$\int_0^1 \phi_3 \phi_3 dt = 1 \quad \int_0^1 \phi_3 \phi_1 dt = 0 \quad \int_0^1 \phi_3 \phi_2 dt = 0$$

$$\int_0^c (bt + a)(bt + a) dt + \int_c^1 (bc + a - b(t - c))(bc + a - b(t - c)) dt = 1$$

$$\int_0^c (bx + a) 1 dt + \int_c^1 (bc + a - b(x - c)) 1 dt = 0$$

$$\int_0^c (bx + a)(2\sqrt{3}t - \sqrt{3}) dt + \int_c^1 (bc + a - b(x - c))(2\sqrt{3}t - \sqrt{3}) dt = 0$$



Linear segments

6. Orthonormalize

$$-2 \cdot b^2 \cdot c^3 + -2 \cdot b \cdot (-2 \cdot b + a) \cdot c^2 + 2 \cdot b \cdot (-b + 2 \cdot a) \cdot c + a^2 - a \cdot b + \frac{1}{3} \cdot b^2 = 1$$

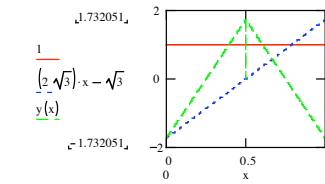
$$\left(-c^2 + 2 \cdot c - \frac{1}{2}\right) \cdot b + a = 0$$

$$\frac{-1}{12} \cdot \sqrt{3} \cdot b \cdot (2 \cdot c - 1 - \sqrt{3}) \cdot (2 \cdot c - 1 + \sqrt{3}) \cdot (2 \cdot c - 1) = 0$$

$$c = \frac{1}{2} \quad \frac{1}{4}b + a = 0 \quad a = -\frac{1}{4}b$$

$$a = -\sqrt{3} \quad b = 4\sqrt{3} \quad c = \frac{1}{2}$$

$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \leq \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$



Linear segments

$$\phi_1 = 1$$

$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$

$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \leq \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$

Decomposition via the “Q” basis

$$\phi_1 = 1 \quad 0 \leq t \leq 1$$

$$\phi_2 = 2\sqrt{3}t - \sqrt{3}$$

$$\phi_3 = \begin{cases} 4\sqrt{3}t - \sqrt{3} & t \leq \frac{1}{2} \\ (2\sqrt{3}t + \sqrt{3}) - 4\sqrt{3}(t - \frac{1}{2}) & t > \frac{1}{2} \end{cases}$$

$$x(t) = \cos(\pi t)$$

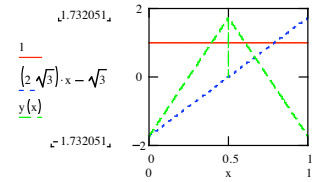
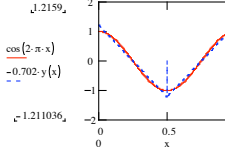
$$x(t) = \sum_k A_k \phi_k(t) \quad \text{where} \quad A_k = \int_a^b x(t) \phi_k^*(t) dt$$

$$A_1 = \int_0^1 x(t) \phi_1 dt \quad A_2 = \int_0^1 x(t) \phi_2 dt$$

$$A_1 = \int_0^1 \cos(2\pi t) dt \quad A_2 = \int_0^1 \cos(2\pi t) (2\sqrt{3}t - \sqrt{3}) dt$$

$$A_1 = 0$$

$$A_2 = 0$$



Linear segments

$$A_3 = \int_0^1 x(t) \phi_3 dt$$

$$A_3 = \int_0^{\frac{1}{2}} \cos(\pi t) (4\sqrt{3}t - \sqrt{3}) dt$$

$$+ \int_{\frac{1}{2}}^1 \cos(\pi t) (2\sqrt{3}t + \sqrt{3} - 4\sqrt{3}(t - \frac{1}{2})) dt$$

$$A_3 = -0.702$$

$$x(t) = -0.702 \phi_3(t)$$

Decomposition via the “Q” basis

$$x(t) = \sin\left(\frac{\pi}{2}t\right) \quad 0 \leq t \leq 1$$

$$x(t) = \sum_k A_k \phi_k(t) \quad \text{where} \quad A_k = \int_0^1 x(t) \phi_k^*(t) dt$$

$$A_1 = \int_0^1 x(t) \phi_1 dt \quad A_2 = \int_0^1 x(t) \phi_2 dt$$

$$A_1 = \int_0^1 \sin\left(\frac{\pi}{2}t\right) dt \quad A_2 = \int_0^1 \sin\left(\frac{\pi}{2}t\right) (2\sqrt{3}t - \sqrt{3}) dt$$

$$A_1 = 0.673$$

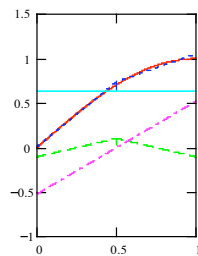
$$A_2 = 0.301$$

$$A_3 = \int_0^1 x(t) \phi_3 dt$$

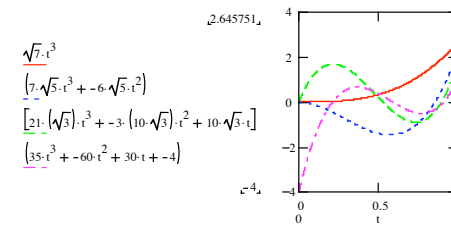
$$A_3 = \int_0^{\frac{1}{2}} \sin\left(\frac{\pi}{2}t\right) (4\sqrt{3}t - \sqrt{3}) dt$$

$$+ \int_{\frac{1}{2}}^1 \sin\left(\frac{\pi}{2}t\right) (2\sqrt{3}t + \sqrt{3} - 4\sqrt{3}(t - \frac{1}{2})) dt$$

$$A_3 = 0.06$$



Orthonormal Cubic Splines

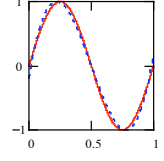
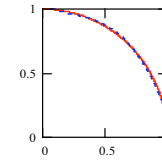
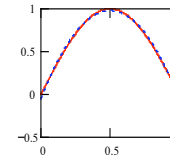
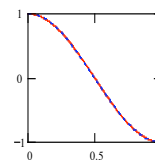


$$\cos(\pi \cdot t)$$

$$\sin(\pi \cdot t)$$

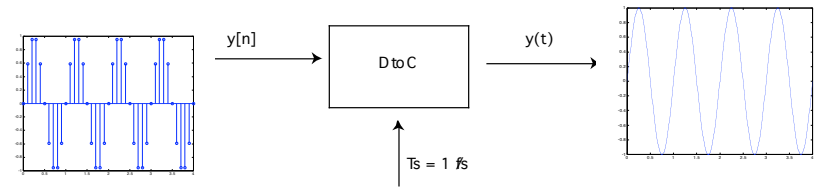
$$\sqrt{1-t^2}$$

$$\sin(2 \cdot \pi \cdot t)$$



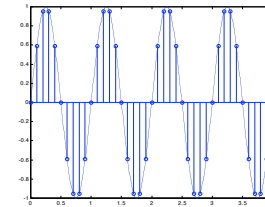
Sampling and Aliasing

D/A Conversion



Ideal discrete to continuous converter for sample spacing T_s

Interpolates discrete samples to form a continuous signal

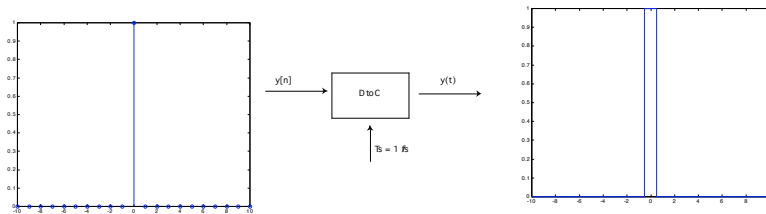


D/A Conversion

Pulse

For each sample $y[n]$, a pulse $p(t)$ is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$



$$y(t) = p(t) = \begin{cases} 1 & -T_s/2 < t \leq T_s/2 \\ 0 & \text{otherwise} \end{cases}$$

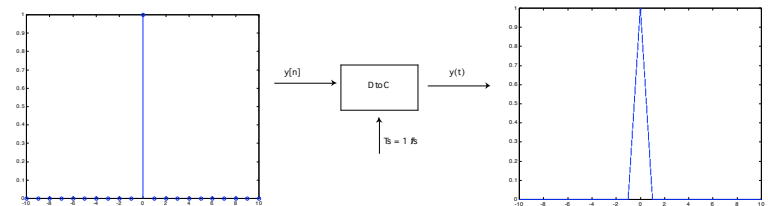
Different pulse shapes produce different D-to-C interpolations

D/A Conversion

Pulse

For each sample $y[n]$, a pulse $p(t)$ is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$



$$y(t) = p(t) = \begin{cases} 1 - |t|/T_s & -T_s \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases}$$

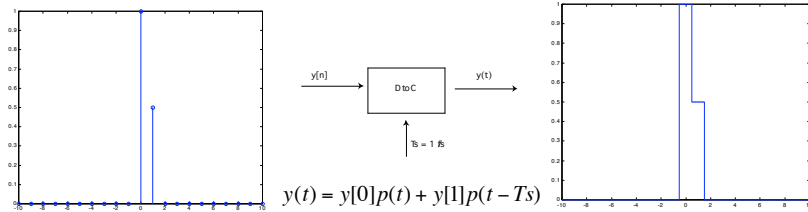
Different pulse shapes produce different D-to-C interpolations

D/A Conversion

Pulse

For each sample $y[n]$, a pulse $p(t)$ is produced

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$



$$y(t) = 1p(t) + 0.5p(t-1)$$

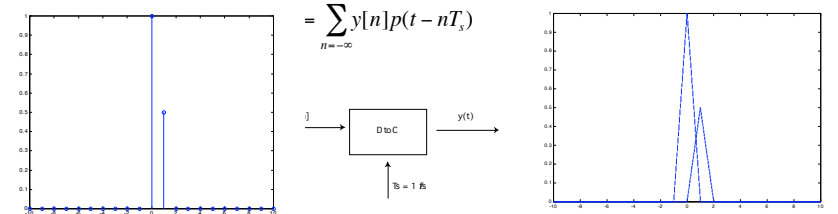
$$p(t) = \begin{cases} 1 & -T_s/2 \leq t \leq T_s/2 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} 1(0) + 0.5(0) = 0 & t < -1/2 \\ 1(1) + 0.5(0) = 1 & -1/2 < t < 1/2 \\ 1(0) + 0.5(1) = 0.5 & 1/2 < t < 3/2 \\ 1(0) + 0.5(0) = 0 & 3/2 < t \end{cases}$$

D/A Conversion

Pulse

For each sample $y[n]$, a pulse $p(t)$ is produced



$$y(t) = y[0]p(t) + y[1]p(t-T_s)$$

$$y(t) = 1p(t) + \frac{1}{2}p(t-T_s)$$

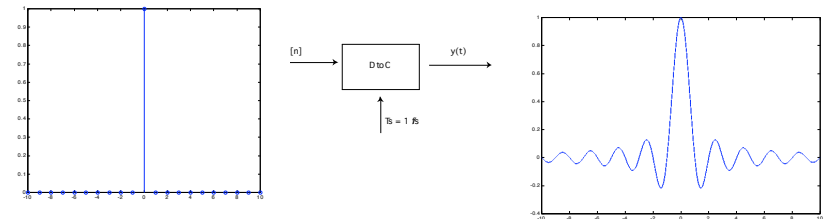
$$y(t) = ap(t) + bp(t-T_s)$$

$$p(t) = \begin{cases} 1 - |t|/T_s & -T_s \leq t \leq T_s \\ 0 & \text{otherwise} \end{cases}$$

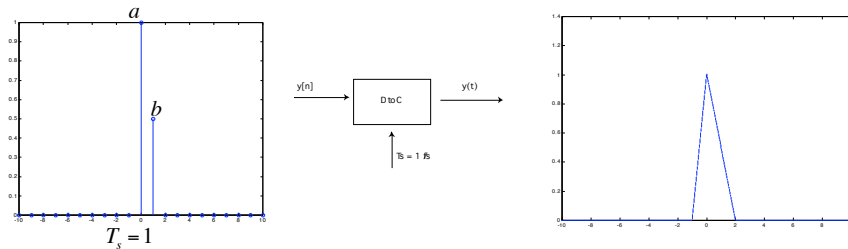
$$y(t) = a \begin{cases} 0 & t < -T_s \\ 1 + t/T_s & -T_s \leq t < 0 \\ 1 - t/T_s & 0 \leq t < T_s \\ 0 & T_s \leq t < 2T_s \\ 0 & t > 2T_s \end{cases} + b \begin{cases} 0 & t < -T_s \\ 0 & -T_s \leq t < 0 \\ 1 + (t - T_s)/T_s & 0 \leq t < T_s \\ 1 - (t - T_s)/T_s & T_s \leq t < 2T_s \\ 0 & t > 2T_s \end{cases}$$

D/A Conversion

Ideal



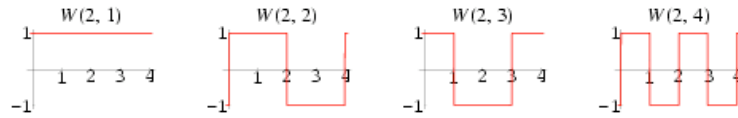
$$y(t) = p(t) = \begin{cases} \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} & \text{for } -\infty < t < \infty \\ \sin c\left(\frac{\pi t}{T_s}\right) \end{cases}$$



$$y(t) = \begin{cases} 0 & t < -T_s \\ 1 + t/T_s & -T_s \leq t < 0 \\ a(1 - t/T_s) + b(1 + (t - T_s)/T_s) & 0 \leq t < T_s \\ 1 - (t - T_s)/T_s & T_s \leq t < 2T_s \\ 0 & t > 2T_s \end{cases}$$

$$y(t) = \begin{cases} 0 & t < -T_s \\ 1 + t/T_s & -T_s \leq t < 0 \\ a + (b - a)t/T_s & 0 \leq t < T_s \\ 1 - t/T_s & T_s \leq t < 2T_s \\ 0 & t > 2T_s \end{cases} \quad \text{linear interpolation}$$

Walsh functions



$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \begin{matrix} \text{normal} \\ \text{orthogonality condition} \end{matrix}$$

$$\int_0^4 \phi_1 \phi_1^* dt = \sum_1^4 1 \cdot 1 = 4 \quad \text{not normal}$$

$$\int_0^4 \phi_2 \phi_2^* dt = (1 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 1 + (-1) \cdot (-1)) = 4 \quad \text{not normal}$$

$$\int_0^4 \phi_1 \phi_2^* dt = 1 \cdot 1 + 1 \cdot 1 + (1 \cdot (-1)) + (1 \cdot (-1)) = 1 + 1 - 1 - 1 = 0 \quad \text{orthogonal}$$

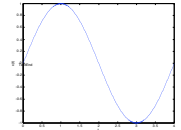
$$\int_a^b \phi_j \phi_k^* dt = \begin{cases} \lambda_k = 4 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Walsh functions



$$x(t) = t \quad 0 \leq t \leq 4$$

$$x(t) = \sum_k A_k \phi_k(t) \quad \text{where} \quad A_k = \frac{1}{\lambda_k} \int_0^4 x(t) \phi_k^*(t) dt = \frac{1}{4} \int_0^4 x(t) W(2, k+1) dt$$



$$A_0 = \frac{1}{4} \int_0^4 t \cdot 1 dt = \frac{t^2}{2} \Big|_0^4 = \frac{1}{4} \cdot 8 = 2$$

$$A_1 = \frac{1}{4} \int_0^2 t \cdot 1 dt + \frac{1}{4} \int_2^4 t \cdot (-1) dt = \frac{1}{4} \left[\frac{t^2}{2} \Big|_0^2 - \frac{t^2}{2} \Big|_2^4 \right] = \frac{1}{4} (2 - 6) = -1$$

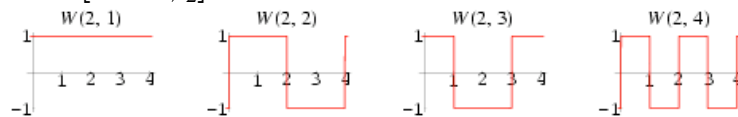
$$A_2 = \frac{1}{4} \int_0^1 t \cdot 1 dt + \frac{1}{4} \int_1^2 t \cdot (-1) dt + \frac{1}{4} \int_2^3 t \cdot 1 dt + \frac{1}{4} \int_3^4 t \cdot (-1) dt = \frac{1}{4} \left(\frac{t^2}{2} \Big|_0^1 - \frac{t^2}{2} \Big|_1^2 + \frac{t^2}{2} \Big|_2^3 - \frac{t^2}{2} \Big|_3^4 \right) = \frac{1}{4} \left(\frac{1}{2} - \left[\frac{4}{2} - \frac{1}{2} \right] + \left[\frac{9}{2} - \frac{4}{2} \right] - \left[\frac{16}{2} - \frac{9}{2} \right] \right) = 0$$

$$A_3 = \frac{1}{4} \int_0^{0.5} t \cdot 1 dt + \frac{1}{4} \int_{0.5}^1 t \cdot (-1) dt + \frac{1}{4} \int_1^{1.5} t \cdot 1 dt + \frac{1}{4} \int_{1.5}^2 t \cdot (-1) dt + \frac{1}{4} \int_2^{2.5} t \cdot 1 dt + \frac{1}{4} \int_{2.5}^3 t \cdot (-1) dt + \frac{1}{4} \int_3^{3.5} t \cdot 1 dt + \frac{1}{4} \int_{3.5}^4 t \cdot (-1) dt = \frac{1}{4} \left(\frac{t^2}{2} \Big|_0^{0.5} - \frac{t^2}{2} \Big|_{0.5}^1 + \frac{t^2}{2} \Big|_1^{1.5} - \frac{t^2}{2} \Big|_{1.5}^2 + \frac{t^2}{2} \Big|_2^{2.5} - \frac{t^2}{2} \Big|_{2.5}^3 + \frac{t^2}{2} \Big|_3^{3.5} - \frac{t^2}{2} \Big|_{3.5}^4 \right) = \frac{1}{4} \left(\frac{1}{8} - \left[\frac{1}{2} - \frac{1}{8} \right] + \left[\frac{9}{8} - \frac{1}{2} \right] - \left[\frac{4}{2} - \frac{9}{8} \right] + \left[\frac{25}{8} - \frac{4}{2} \right] - \left[\frac{9}{2} - \frac{25}{8} \right] + \left[\frac{49}{8} - \frac{9}{2} \right] - \left[\frac{16}{2} - \frac{49}{8} \right] \right) = -\frac{1}{2}$$

Walsh functions

$$x(t) = \sum_k A_k \phi_k(t)$$

$$A_k = \left[2, -1, 0, -\frac{1}{2} \right]$$



$$[2 \ 2 \ 2 \ 2] + [-1 \ -1 \ 1 \ 1] + [0 \ 0 \ 0 \ 0] + [-0.5 \ 0.5 \ -0.5 \ 0.5] = [0.5 \ 1.5 \ 2.5 \ 3.5]$$

