MAS.836 Sensor Technologies for Interactive Environments
Problem Set 2: Time-Varying Circuit Analysis

The purpose of this problem set is to introduce capacitors, which are dependent on time-varying signals, $v(t)$. The op amp circuits that were introduced in problem set 1 are also dependent on time. This problem set will also explore the concept of op amp circuits with positive feedback.

Complex Frequency Analysis

Calculations involving time-dependent circuits are greatly simplified through the use of complex math. By using these complex numbers, to represent signals and impedances, we can treat capacitors and inductors the same way as resistors; and sinusoidal signals the same as constant voltages. This representation of sinusoidal signals is called a *phasor* [2].

Two important symbols will appear when analyzing time-dependent circuits. The first is the constant $j = \sqrt{-1}$. You may note that this constant is represented with i by mathematicians, but since electrical engineers use i to represent currents, we use j instead.

The second is the angular frequency in radians per second, which we'll represent with ω. Remember that the relationship between the angular frequency ω and the frequency in Hertz, f, is as follows:

$$\omega = 2\pi f \tag{1}$$

Recall that complex numbers are vectors that have an imaginary component (the term(s) multiplied by j) and a real component (everything else.) If we have the complex number $a + jb$, a is the real component and b is the imaginary component. We can represent complex numbers as vectors on a plane, with the real component along the X-axis and the imaginary component along the Y-axis [3]. The *magnitude* and *phase angle* of this vector have physical significance.

For our complex signal $v(t) = a + jb$, the *magnitude* $|v_{out}(t)|$ is the length of the vector:

$$|v(t)| = \sqrt{a^2 + b^2} \tag{2}$$

and the *phase* $\angle v(t)$ of the signal is the angle between the vector and the X-axis:

$$\angle v(t) = \arctan \frac{b}{a} \tag{3}$$

The magnitude, as the name suggests, is physically significant as the amplitude of the signal if you were to measure it on an oscilloscope. The phase is less relevant to this class, but it corresponds to a time delay of the signal: a signal that has a phase of $90°$ at the output has been delayed by $1/4$ period; a phase shift of $180°$ is the same as inverting the signal.

Notice that we can represent phasors in two different, but equivalent ways: either as complex numbers, or a phase and a magnitude. We can go between these two representations with equations 2 and 3. In the complex representation, we can easily do mathematical operations on the phasors, like applying Ohm's law, calculating series impedances, etc. The phase/magnitude representation tells us more about the physical properties of our signals.

For more information on complex math and phasors, refer to *The Art of Electronics* [1], pages 28-41. Complex math is also covered in many college-level math texts.

Capacitor Impedance

Using our knowledge of phasors, we can think of capacitors as frequency-dependent resistors with impedance shown in equation 4. Remember that *impedance* is just like resistance, but with j and ω it can also handle time-dependent signals.

$$Z_c = \frac{1}{j\omega C} \tag{4}$$

As the frequency increases, the impedance decreases. At very high frequencies ($\omega \to \infty$) the impedance (again, think resistance) goes to zero, so a capacitor is like a $0\ \Omega$ resistor, or a short circuit, for a high-frequency signal. At low frequencies ($\omega \to 0$) the capacitor is like an infinite resistor, or an open circuit.

Ohm's Law still applies with complex impedance, Z, replacing resistance, R:

$$v_1 = i_1 \cdot Z_c = i \cdot \frac{1}{j\omega C} \tag{5}$$
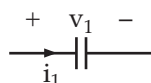
Figure 1: Symbol for a capacitor, with current and voltage labels

Non-Ideal Op Amp Model

The ideal op amp model presented in Problem Set 1 is useful for analyzing many circuits, but does not hold true under all circumstances. The op amp's voltage output is limited by the power supply of the op amp. The op amp cannot create a V_{out} that is larger than the positive supply $+V_s$, nor can it create a V_{out} which is lower than the negative supply $-V_s$. If the output of the op amp tries to swing outside of the supply voltages, it merely *saturates* (that is, the output remains at the maximum possible voltage, either at $+V_s$ or $-V_s$.) Op amps that saturate to the maximum and minimum supply voltages are called *rail-to-rail* op amps. Some op amps are only able to saturate within ≈ 3 volts of the supply voltages! Be sure to carefully check the limitations of op amps that you select.
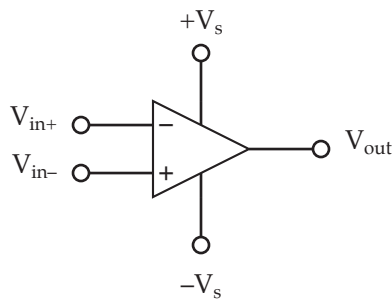
Figure 2: Operational amplifier with supply voltage notation

Without negative feedback, V_{in-} cannot be assumed to be the same as V_{in+}. Instead, the relationship $V_o = A_v(V_{in+} - V_{in-})$ must be used to calculate the output voltage. When the op amp design is configured for positive feedback, this calculation simplifies to:

$$V_o = \begin{cases} +V_s : \text{when } v_{in+} > v_{in-} \\ -V_s : \text{when } v_{in+} < v_{in-} \end{cases} \tag{6}$$

Any difference between V_{in+} and V_{in-} will cause the op amp to saturate because A_v is very large.

1   Decibels and Non-Ideal Op Amps

A decibel (dB) is defined as the power ratio between two signals. Decibels are useful in cases where the power ratio may span many decades, as a decibel scale has a logarithmic response. We define decibels as follows:

$$dB = 10 \cdot \log_{10} \frac{P_1}{P_2} \tag{7}$$

For example:

$$40 \, dB = 10 \cdot \log_{10} \frac{1 \times 10^6}{100} \tag{8}$$

In many cases, we are interested in the ratio of voltages or currents, rather than the ratio of powers. Ratios of voltages or currents may still span over many decades, thus making decibel representation useful. For consistency, the voltages must be converted into powers. Recall that:

$$P = \frac{v^2}{R} \tag{9}$$

If we hold R constant and substitute equation 9 into equation 8, the resistances cancel and the squares on the voltage terms come outside the logarithm as a factor of two:

$$dB = 20 \cdot \log_{10} \frac{v_1}{v_2} \tag{10}$$

Similarly, for currents:

$$dB = 20 \cdot \log_{10} \frac{i_1}{i_2} \tag{11}$$

Questions

The large-signal voltage gain A_v of the op amp you are using in the lab assignments is given in the datasheet as 100 dB.

1. What is A_v in $V/\mu V$? Note that most datasheets either list A_v in dB or $V/\mu V$.

2. If you are using this op amp in a circuit with $+V_s = 5$ V and $-V_s = 0$ V, what is the maximum input differential $(v_{in+} - v_{in-})$ that it can handle before it saturates?

2 Comparators

The following circuit is known as a comparator. It is useful for performing single-bit analog-to-digital (A/D) conversion, or informing some other part of the circuit whether or not a circuit has crossed a certain threshold. The input may be placed into either the inverting or non-inverting input. Use the circuit in figure 3 and the plot in figure 4 to answer the questions.
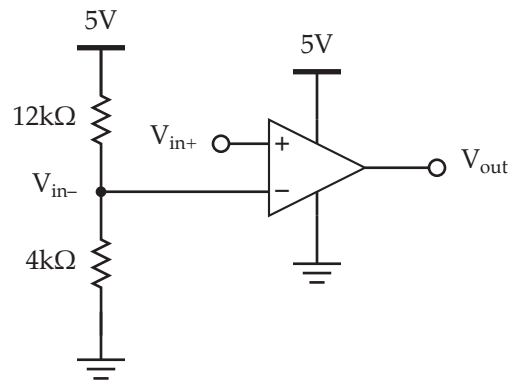

Figure 3: Comparator circuit
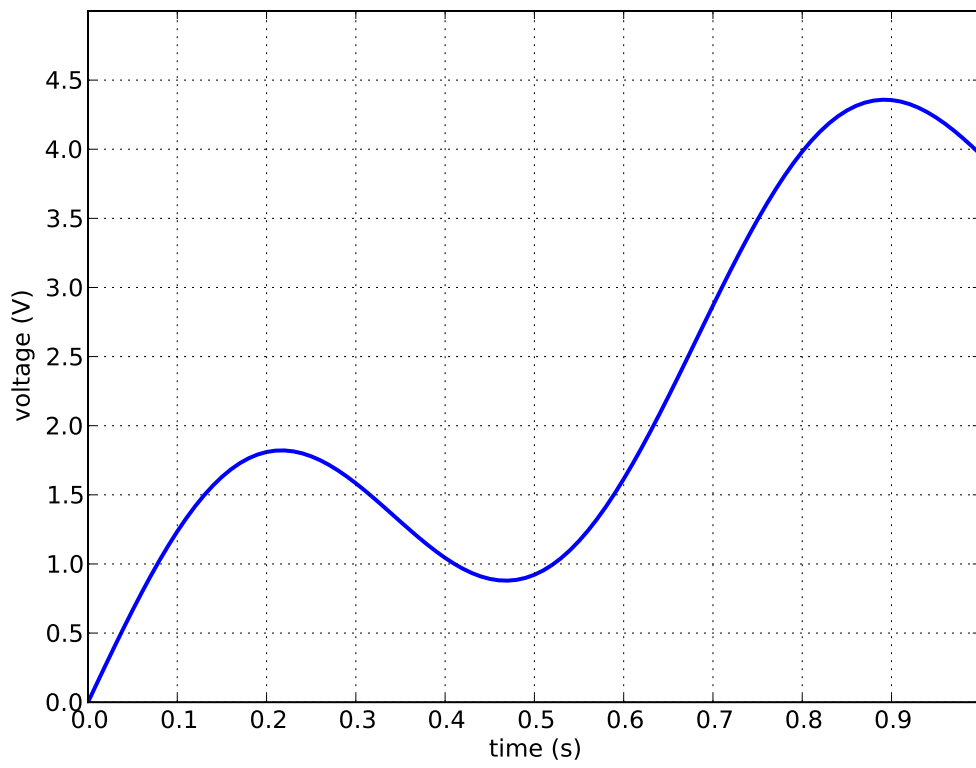
Figure 4: Plot of v_{in+} over time for questions

Questions

1. What is the value of the voltage at v_{in-}?

2. Will v_{in-} vary with time?

3. Plot v_{in-} as a function of time on figure 4.

4. Plot v_{out} as a function of time on figure 4.

3 Schmitt Triggers

The previous op amp circuit is susceptible to false triggering due to noise on the input signal (note your answer to problem 1, part 2.) To eliminate this, you can add a resistor to provide positive feedback that pulls the reference voltage in one direction or another. This configuration, shown below, is called a Schmitt trigger input, and it creates hysteresis to debounce the input signal. Use the circuit below to answer the questions (note that the op amp's inputs have been switched from the circuit in problem 2.)
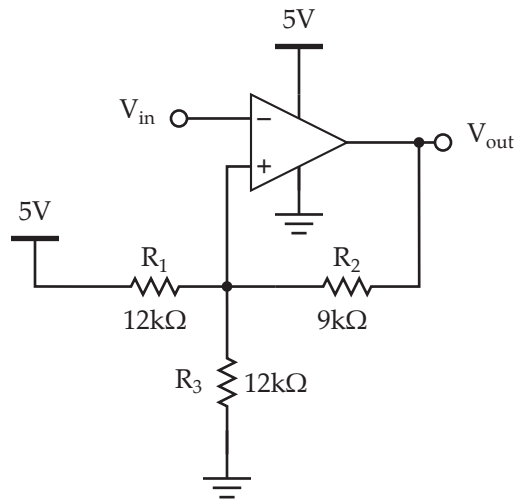
Figure 5: Schmitt trigger circuit
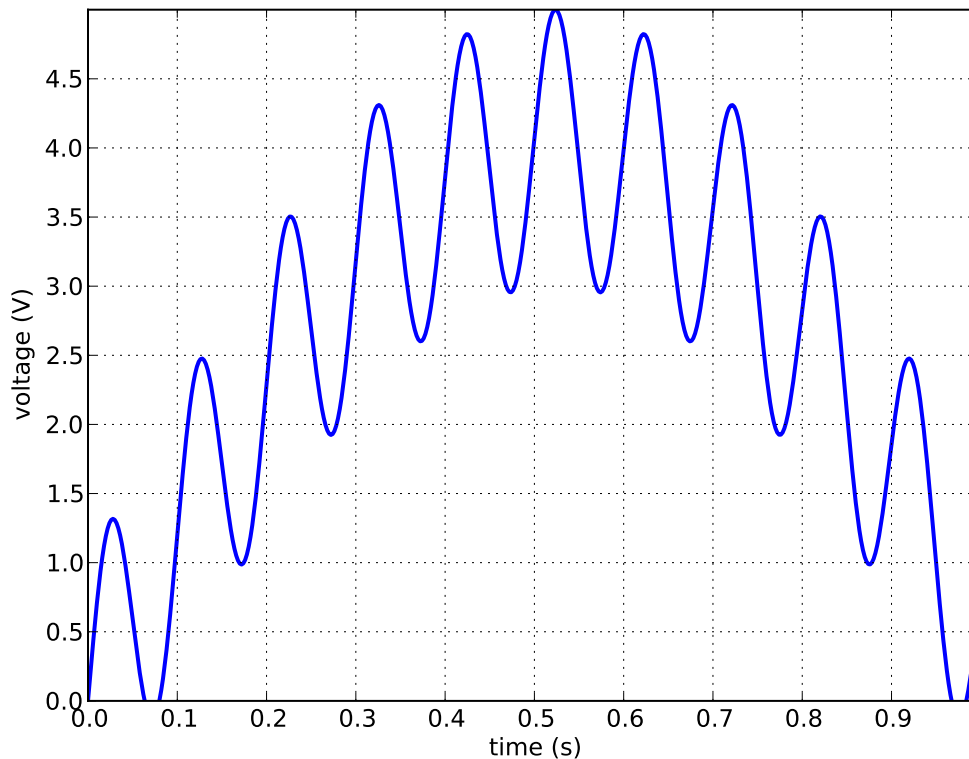
Figure 6: Plot of v_{in} over time for questions

Questions

1. If $V_{out} = 5$ V, i.e. is saturated to $+V_s$, what is the value of v_{in+}?

2. If $V_{out} = 0$ V, i.e. is saturated to $-V_s$, what is the value of v_{in+}?

3. Using dashed lines, plot both values of v_{in+} from (1) and (2) as constant voltages versus time on the plot in figure 6.

4. If $V_{in} = 0$ V, what is V_{out}?

5. Given the plot of V_{in} in figure 6, plot V_{out} and v_{in+} as a function of time.

4   Passive Low-Pass Filters

The following circuit is an RC low-pass filter: it uses a resistor and a capacitor to pass low frequencies and attenuate high frequencies. The defining characteristic of an RC filter is its cutoff frequency ω_c, which is when the circuit transitions from passing one set of frequencies to attenuating another. This happens at the point where the magnitude of the real impedance provided by the resistor equals the magnitude of the complex impedance presented by the capacitor ($Z_c = \frac{1}{j\omega C}$).
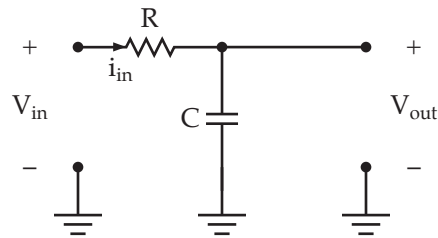

Figure 7: RC low-pass filter

1. What is the input impedance $\frac{v_{in}}{i_{in}}$ at DC (0 Hz)?

2. What is the input impedance at infinite frequency?

3. What is the transfer function $\frac{v_{out}}{v_{in}}$ of this circuit as a function of R, C, j, and ω?

4. What is the cutoff frequency ω_c of this circuit in radians/second?

5. What is the magnitude, as described in the complex frequency analysis section, of your result from (3) at the cutoff frequency (4)?

6. What is the value of V_{out} at DC?

7. What is the value of V_{out} at infinite frequency?

8. For high frequencies, such that $\omega \gg \frac{1}{RC}$, what is the ratio, in dB, of the amplitude of V_{out} at ω to the amplitude of V_{out} at 10ω? (This is called the roll-off.)

5 RC High-Pass Filters

The following circuit is called an RC high-pass filter: it passes high frequencies and attenuates low frequencies. It follows the same rules as described in problem 4.
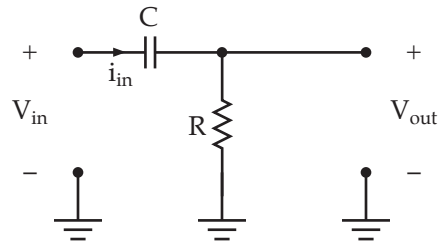

Figure 8: RC high-pass filter

Questions

1. What is the input impedance $\frac{v_{in}}{i_{in}}$ at DC (0 Hz)?

2. What is the input impedance at infinite frequency?

3. What is the transfer function $\frac{v_{out}}{v_{in}}$ of this circuit as a function of R, C, j, and ω?

4. What is the value of V_{out} at DC?

5. What is the value of V_{out} at infinite frequency?

6. For low frequencies, such that $\omega \ll \frac{1}{RC}$, what is the ratio, in dB, of the amplitude of V_{out} at ω to the amplitude of V_{out} at 10ω?

7. If $R = 2.2\ \text{k}\Omega$ and $C = 2\ \text{nF}$, what is the cutoff frequency f_c in Hertz?

6 Multiple-Feedback Bandpass Filters

In this problem you will design a Multiple-Feedback (MFB) bandpass filter. Review chapter 16 of *Op Amps for Everyone* [4]. Note that to bias an MFB bandpass op amp's output, the v_+ terminal can be set to the middle voltage range with a voltage divider. See "How to Bias an Op Amp" [5].

Design an MFB bandpass filter with one op amp and the following parameters. Check your design, which you will calculate manually, with a filter design tool.

- uses feedback capacitors $C = 2.2$ nF
- has a center frequency $f_m = 2.5$ kHz
- has a gain at the center frequency of $-A_m = 3$
- has a bandwidth of 500 Hz
- runs off a single 5 V supply (i.e. $+V_s = 5$ V and $-V_s = 0$ V)
- has an output which is centered around 2.5 V.

Include a calculation of the quality factor Q for your filter.

References

[1] P. Horowitz and W. Hill, *The Art of Electronics.* New York: Cambridge University Press, 1989

[2] Weisstein, Eric W. "Phasor." From MathWorld—A Wolfram Web Resource.
`http://mathworld.wolfram.com/Phasor.html`

[3] Weisstein, Eric W. "Complex Plane." From MathWorld—A Wolfram Web Resource.
`http://mathworld.wolfram.com/ComplexPlane.html`

[4] R. Mancini, *Op Amps For Everyone: Design Reference.* Dallas, Texas: Texas Instruments, 2002.

[5] M. Feldmeier, "How to Bias an Op Amp", Cambridge, Massachusetts: MIT Media Lab, 2007. `http://resenv.media.mit.edu/classes/MAS836/bias.pdf`

[6] *Active Filter Synthesis Design Tool*, Analog Devices.
`http://www.analog.com/en/content/filter_wizard/fca.html`